
THE JOURNAL

OF THE

Indian Mathematical Society.

EDITED BY

M. T. NARANIENGAR, M.A.,

Hon. Joint Secretary

WITH THE CO-OPERATION OF

Prof. R. P. PARANJPYE, M.A., B.Sc., Prof. A. C. L. WILKINSON, M.A., F.R.A.S.

and others

VOL. VI.

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WANTED.

Solution to the Following :—

Questions.—468, 474, 476, 477, 478, 480, 489, 495, 508, 518, 526, 530
533, 534, 535, 538, 539, 540, 541, 545, 546.

ERRATA.

Page.	Line.	For	Read
4	2	sin OD ...	... sin MD.
5	4	$\sin x$ ...	... $\sin \times$.
6	14	G	... G_1.
	12	cot S − A : cot S − B : cot S − C.	cot(S − A) : cot(S − B) : cot(S − C).
	20	A_1N	... A_1M.
10	15	$n_1 A_2 B_3$ ...	... $n_1 . A_2 B_1$,
,,	16	$n_2 A_2 B_b$ ...	... $n_2 . A_2 B_2$.
11	37	A_1, A_2. ...	... $A_1, A_2, \ldots\ldots$
15	12	∠BTB ...	... ∠BTC.
17	12	$x\gamma_1{}^2, + y\gamma_1{}^2$, ...	... $x_1{}^2 + y_1{}^2$.
,,	19	$\Sigma u' \dfrac{\partial + (u_1, u_1\theta)}{\partial u_1'}$	... $\Sigma u' \dfrac{\partial f(u_1, u_1')}{\partial u_1'}$
,,	20	$2v_1{}^5$	... $2v_1'$.
18	2	$(x_3 - x_1)^2 + \ldots\ldots$	... $(x_1 - x_2)^2 + (y_1 - y_2)^2 + (x_3 - x_1)^2 + (y_3 - y_1)^2 - (x_2 - x_3)^2 - (y_2 - y_3)^2$.
19	8	$(a^2 - \lambda)^2$ ...	... $(a^2 - \lambda^2)$.
,,	15	$\dfrac{X^2}{a^2 - \lambda^2} =$ ...	... $\dfrac{X^2}{a^2 - \lambda^2} +$
,,	17	$\dfrac{\lambda_4{}^2 \lambda_2{}^2 \lambda_3{}^2}{a^2 b^2}$ ...	... $\dfrac{\lambda_1{}^2 \lambda_2{}^2 \lambda_3{}^2}{a^2 b^2}$.
	28	$\sqrt{\dfrac{-(a^2 - \alpha^2{}_1) \text{ \&c.}}{ab}}$	$\sqrt{\dfrac{-(a^2 - \alpha_1{}^2)(a^2 - \alpha_2)^2(a^2 - \alpha_3{}^2)}{ab}}$
20	19	$2b^2\sqrt{a^4 - a^2{}_{-6}b^2 + b^4}$	... $2b^2\sqrt{a^4 - a^2b^2 + b^4}$.
21	29	it	... its.
24	29	$s_3 + s_1 - s_3)(1 - s_2)$	... $s_3 + (s_1 - s_3)(1 - s_2)$.
26	4	De^y	... De^{-y}.
,,	5	α	... ∞.
27	10	$\left(\Gamma \dfrac{2}{n}\right)$ ...	... $\left(\Gamma \dfrac{2}{n}\right)^2$

Page.	Line.	For	Read
28	3	$\propto$	∞.
	7	$d_1+d_2+d_3+d_n$ ...	$d_1+d_2+\ldots+d_n$.
29	7	cos. $2bz$	cos $2bz$.
,,	8	$\sqrt{\alpha.x}$	$\sqrt{\alpha.x}$.
,,	9	$\sqrt{\gamma}$	$\sqrt{\beta}$.
,,	14	-6^2	$-b^2$.
,,	,,	-46^2	$-4b^2$.
30	14	$p\rho-\gamma$	$p\rho-r$.
31	4	eithers	either.
,,	6	axe	axes.
32	11	$\sec^2\beta^4$	$\sec^2\beta$.
33	13	$y_u^{-1}+y^{n-3}+\ldots y^{-1+3}+y^{-n+1}$.	$y^{n-1}+y^{n-2}+\ldots+y^{-n+3}+y^{-n+1}$.
35	3	$V\pi$	$\sqrt{\pi}$.
36	13	$x \sin^1 x$...	$x \sin^{-1} x$.
,,	19	$\sqrt{1-x^2}$	$\sqrt{1-x^2}$.
37	15	foc	foci.
38	3	line	lie.
44	*last*	or	for.
45	19	expreesed	expressed.
,,	22	G_1xr	G_1xz.
48	19	$b^2\mu$	$b\mu^2$.
51	10	(x^2+y^2-c')	$(x^2+y^2-c')^2$.
52	14	$(y+\beta)^2$	$(y-\beta)^2$.
,,	19	$(b+\lambda_2)(b+\lambda_2)$... ...	$(b+\lambda_1)(b+\lambda_3)$.
,,	20	$(b+\lambda^2)$	$(b+\lambda_2)$.
,,	27	$b+\lambda_{23}$	$b+\lambda_3$.
53	17	$y=z$	$y=x$.
54	7	λ^3	λ_3.
,,	10	λ^3	λ_3.
58	17	$P\hat{O}P$	$P\hat{O}B_n$.
59	2	POB′	POB_r.
63	4	$\frac{ON}{OP}=\frac{OP}{OT}=\frac{OQ}{OT}$ ...	$\frac{ON}{Op}=\frac{Op}{OT}=\frac{Oq}{OT}$
65	8	1	I.
65	13	1	I.
68	14	$\overline{4}$	$\frac{b^2}{4}$.

Page.	Line.	For	Read
68	16	$\int_{\frac{1}{4}b^2}^{-1}$	... $\int_{\frac{1}{4}b^2}^{1}$.
70	3	$2b\gamma\gamma^2$	... $2b\gamma\gamma_2$.
71	5	b°	... b_{o}.
73	5	$+0$	... $=0$.
77	1	similaar ...	... similar.
78	23	$a^1x^1+2h^1xy+b\mu^2$	... $a'x^2+2h'xy+b'y^2$.
79	*last*	$\mathrm{co}_{ax}{}^2$	... $\cos ax^2$.
83	21	in tuition ...	... intuition.
90	11	lead	... led.
91	*last*	to	... to a.
94	35	emperical ...	... empirical.
101	21	$A^{n-1}{}_r{}^{n-1}$ ...	... $A_{n-1}r^{n-1}$.
103	7	$\lfloor n-1$... ...	... $\underline{\lfloor n-1}$.
"	21	$x\,(1+x^2)n$ ...	... $x\,(1+x^2)^n$.
"	23	$\underline{\lfloor n}-1$	... $\underline{\lfloor n-1}$
112	*last*	(M I	... XM.
115	6	criclee ...	... circle.
"	11	finit	... finite.
118	16	$(xyz_1)^2$ ...	... $(xyz)^2$.
125	1	hyperbol ...	... hyperbola.
136	4	Basseet ...	... Basset.
138	4	AB'C ...	... AB'C'.
140	1	raddily ...	... readily.
"	"	an	... and.
"	8	AQ AQ' ...	... AQ. AQ'.
145	1	β	... B.
148	8	$2hny$ ...	... $2hxy$.
151	24	$\mid -1, \mid$ ...	... $\mid -1, 1 \mid$.
153	7	1	... I.
156	2	$\cos\,(\tan h^{=}x)$ ...	... $\cos\,(\tan h^{-1}x)$.
"	21	$(-1)A_{2r+1}$ ...	... $(-1)^r A_{2r+1}$.
160	20	tatra— ...	... tetra—
163	9	theh istory ...	... the history.
169	13	d	... d'
178	*last*	et	... meet.
183	7	$\sin^{\frac{1}{n}-1} n\theta$ ...	... $\sin^{\frac{1}{n}-2} n\theta$.

Page.	Line.	For	Read
183	8	$\sin^{1-\frac{1}{n}} n\theta$ ...	... $\sin^{2-\frac{1}{n}} n\theta$.
186	16	cnts ...	... cnt—
189	16	$\int_0^{\sqrt{1-x^2}}$ ...	... $\int_0^{\sqrt{(1-x^2)}}$.
„	„	$\frac{xy^3}{(1-x^2-y^2)}^{\frac{1}{2}}$ ...	... $\frac{xy^3}{(1-x^2-y^2)^{\frac{1}{2}}}$.
190	8	$\sqrt[3]{\cos\ \pi/7}$ ...	... $\sqrt[3]{\cos 4\pi/7}$.
„	15	$\sqrt[3]{\alpha\beta\gamma}$ ...	... $\sqrt[3]{\alpha\beta\gamma}$.
„	24	$\cos\frac{2\gamma\pi}{7}$ ...	... $\cos\frac{2\gamma\pi}{7}$.
191	„	$\sqrt[3]{pq}$ ...	... $\sqrt[3]{pq}$.
193	*last*	$\int_{-\infty}^{\infty}\frac{\sin xy}{x(1+x^2)^3}$	... $\int_{-\infty}^{\infty}\frac{\sin xy}{x(1+x^2)^3}dx$.
„	„	$\int_{-\infty}^{\infty}\frac{\cos xy}{x(1+x^2)^3}$	... $\int_{-\infty}^{\infty}\frac{\cos xy}{x(1+x^2)^3}dx$.
195	3	$A+B+\frac{\pi}{2}0$ ...	... $A+B+\frac{\pi}{2}=0$.
196	1	$M^{\frac{3}{2}}$	... $M^{\frac{3}{2}}$
„	8	$y\sqrt{a^1}=y\sqrt{a^1b^2}-\lambda^{12}$	... $Y\sqrt{a'}=y\sqrt{a'b'-h'^2}$.
„	10	$2hh$	... $2hh'$.
„	12	$\int_{-\infty}^{+}$ ...	... $\int_{-\infty}^{+\infty}$
220	23	$a_np^{(2n-1)}$ ...	... $a_np^{(2n-1)n}$.
221	16	$\frac{172}{15}$	... $\frac{512}{45}$
222	8	AA	... AA′.
228	26	$\frac{d^2y}{dx^2}$	... $\frac{d^3y}{dx^3}$.
229	4	actor	... factor.
229	16	$-3^{-3}/_2$ ...	... $-3z^{-\frac{3}{2}}$.
230	„	easily to ...	... easily seen to.

Page.	Line.	For	Read
233	7	$\left[\int_{0}^{\frac{1}{8}\pi}+\int_{\frac{1}{8}\pi}^{\frac{1}{4}\pi}\right]$ ...	... $\left[\int_{0}^{\frac{\pi}{8}}+\int_{\frac{\pi}{8}}^{\frac{\pi}{4}}\right]e^{-R^2\cos 2\theta}$
235	1	$\frac{\partial\phi}{\partial t}\left(\frac{\partial\phi}{\partial t}+\frac{\partial f}{\partial a}\cdot\frac{\partial a}{\partial t}\right)$ ...	... $\frac{\partial\phi}{\partial t}\left(\frac{\partial f}{\partial t}+\frac{\partial f}{\partial a}\cdot\frac{\partial a}{\partial t}\right)$
„	„	$\frac{\partial j}{\partial}$	... $\frac{\partial f}{\partial t}$.
„	2	$\left(\frac{\partial f}{\partial t}\right)^3$ ...	... $\left(\frac{\partial f}{\partial t}\right)^2$.
„	16	$\cos h_2 n(t+A)$ ...	... $\cosh 2n(t+A)$.
237	19	2735 ...	... 2835.
238	5	in-Fenrbach ...	... in-Feuerbach.
„	6	Fenrbach ...	... Feuerbach.
239	*last line* of Q. 607	...	... $3\,(x^3+etc.)\div(x^2+etc.)$
240	16	$\lvert 2m'$ ...	... $\underline{\lvert 2m'}$.

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CONTENTS.

The Journal is open to contributions from members as well as subscribers. The editors may also accept contributions from others.

Contributors will be supplied, if so desired, with extra copies of their contributions at net cost.

All contributions should be written legibly on one side only of the paper, and all diagrams should be given in separate slips.

All communications intended for the Journal should be addressed to the Hony. Joint Secretary, M. T. NARANIENGAR, M.A., Mallesvaram, Bangalore.

All business communications regarding the Journal should be addressed to the Hony. Asst. Secretary, P. V. SESHU AIYAR, B.A., L.T., 37, Venkatachala Chetty Street, Triplicane, Madras.

Enquiries by intending members and all other communications may be addressed to the Hony. Joint Secretary, D. D. KAPADIA, M.A., B.Sc., 322, Main Street, Poona.

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Vol. VI.] FEBRUARY 1914. [No. I.

PROGRESS REPORT.

The following gentlemen have been elected members of the Society :—

(1) *Mr. S. G. Burrow, B. Sc.*, (London) *A. R. C. Sc.*—Principal and Professor of Physics, Baroda College, Baroda;

(2) *Mr. Shiv Narayan, M. A., B. Sc.* (Punjab), *B. E.* (Union, America), *A.M.I.E.E.*,—Assistant Professor of Electrical Engineering, College of Engineering, Poona;

(3) *Mr. Kalluri Sundara Ramiah B.A., L.T.*—Mathematics Assistant, Town High School, Guntur, (*at concessional rate*).

(4) *Mr. K. S. Srinivasachari, B.A., L.T.*—Teacher, Mahant's High School, Tirupati, (Chittor) (*at concessional rate*).

(5) *Mr. S. V. Venkataraya Sastri M.A., L.T.*—Mathematics Lecturer, Victoria College, Palghat, (*at concessional rate*).

2. The Committee feel pleasure to announce that—

(i) *The Tohoku Mathematical Journal*, edited by Professor T. Hayashi, and published at Tokyo (Japan) will be received in exchange for our Journal;

(ii) *The University of Illinois* will send in exchange for our journal annually, from their Department of Mathematics the Ph. D. theses and other mathematical articles as they are published.

(iii) "*Nature*" will be available for issue every week for the current year, through the courtesy of Mr. Balak Ram, I.C.S.

3. The following books have been received from the Publishers—

(1) *Numerical Trigonometry*—by Messrs. Borchardt & Perrott, 2*s*. 6*d*; London, 1913. (George Bell & Sons).

(2) *A Shorter Algebra*—by Messrs Baker and Bourne, 1913, 2/6. (George Bell & Sons).

(3) *The Calcutta University Calendar* for 1913, Part IV.

(4) *Brief Biography* and *Popular Account* of the Unparalleled Discoveries of Prof. Dr. T. J. J. Sly—by W. L. Webb. Lyma, Mass, U. S. A., 1913 (T. P. Nichols & Son Co.).

(5) *Leçons de Mathematiques Gēnērales*—by Prof. L. Zoretti, with a preface by P. Appell, Paris, 1914, 20 frs. (Gauthier—Villars).

(6) *Nature*—Vols. 88, 89, 90, & 91, as presents, from Mr. Balak-Ram, I. C. S.

POONA, 31*st Jan.*, 1914.

D. D. KAPADIA,
Hon. Joint Secretary.

UNIVERSITY LECTURES IN MADRAS.

We are pleased to note that a course of University lectures twenty one in number, was recently delivered in the Presidency College, Madras, by Mr. E H. Neville, Fellow, of Trinity College, Cambridge. The course began on January the 5th and terminated on February 7th.

For the benefit of those who could attend the lectures for only one week, Mr. Neville gave two lectures on 'THE NATURE OF REAL AND COMPLEX NUMBERS' and two lectures complete in themselves on 'DIFFERENTIAL GEOMETRY.' The remainder of the course was taken up by a detailed treatment of 'MOVING AXES AND DIFFERENTIAL GEOMETRY,' a subject which Mr. Neville has made his own.

The course was well attended and drew an audience not only from the south of India, but also from so far north as Aligarh. Mr. Neville will, it is hoped, issue his lectures in book form during the course of the year.

The Spherical Mean Centre.

By M. T. Naraniengar.

1. Let A_1, A_2 be points on a great circle of a sphere whose distances from an origin O on the great circle are α_1, α_2; and let M be a point determined by the equation $\sin(x-\alpha_1)+\sin(x-\alpha_2)=0$.

Then, evidently, $x=\frac{1}{2}(\alpha_1+\alpha_2)$, so that M is the mid-point of A_1A_2;

and $$\sin\alpha_1+\sin\alpha_2=2\sin x\cos\tfrac{1}{2}A_1A_2 \propto \sin x.$$

The point M is called the *spherical mean centre* of A_1A_2.

2. If A_1, A_2, A_3......be any number of points on a great circle whose distances from O are $\alpha_1, \alpha_2, \alpha_3$......, then their *s. m. c.* is M determined by the equation

$$\sin(x-\alpha_1)+\sin(x-\alpha_2)+\ldots=0,$$

or $$\Sigma\sin(A_1M)=0,$$

so that $$\tan x=\tan OM=\Sigma(\sin\alpha_1)/\Sigma(\cos\alpha_1),$$

and $$\sin\alpha_1+\sin\alpha_2+\sin\alpha_3+\ldots\ldots=\Sigma\sin[x-(x-\alpha_1)]$$
$$=\sin x\Sigma\cos(x-\alpha_1)-\cos x\Sigma\sin(x-\alpha_1)$$
$$=\sin x\Sigma\cos(A_1M)\propto\sin x,$$

since $\Sigma\sin(x-\alpha_1)=0$, and M is fixed relatively to A_1A_2......

3. If A_1, A_2,......be a set of points on a great circle whose *s.m.c.* is M, then

$$\Sigma\cos OA_1\propto\cos OM,$$

where O is any point on the great circle.

For,

$$\Sigma\cos OA_1=\Sigma(\cos\alpha_1)=\Sigma\cos[x-(x-\alpha_1)]$$
$$=\cos x\Sigma\cos(x-\alpha_1),\ \text{since}\ \Sigma\sin(x-\alpha_1)=0,$$
$$=\cos x\Sigma\cos(A_1M)\propto\cos OM.$$

Cor. $\Sigma\cos(OA_1)$ is greatest, when $x=0$; that is when O is at M.

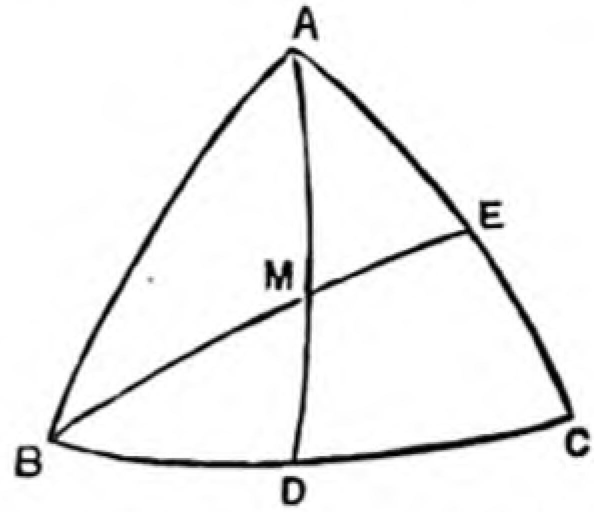
4. Suppose ABC is a spherical triangle and M is a point within the triangle defined by the relation

$$\text{arc MA}+\text{arc MB}+\text{arc MC}=0,$$

where the *plus* sign has to be interpreted as in the case of *vectors*.

Thus to combine arc MB, and arc MC, we may proceed as follows:

Bisect BC at D and join MD. Then the *sum* or *resultant* of arcs MB, MC is X or λ MD, such that sin $X=\lambda$ sin OD, where λ stands for 2 cos $\frac{1}{2}$ BC.



Now, if M be the median centre of a spherical triangle ABC, we know that

$$\sin AM=2 \cos \tfrac{1}{2} a . \sin MD, \text{ (Casey : *Sphl. Trig.* p. 72.)};$$

so that
$$\text{arc } MB+\text{arc } MC=X=\text{arc } AM.$$

Thus the point M defined by the equation

$$\text{arc } MA+\text{arc } MB+\text{arc } MC=0$$

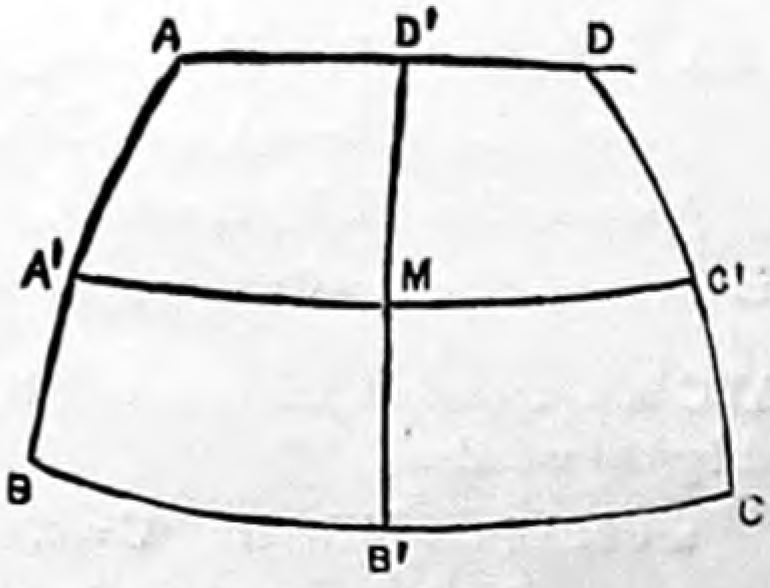
is the centre of medians of the triangle ABC, and may be called the *s. m. c.* of A, B, C.

5. If A, B, C, D......be any number of points on a sphere, the point M defined by the equation

$$\Sigma\,(\text{arc } MA)=0$$

is the *s. m. c.* of the system, and a similar geometrical interpretation of the equation will hold. Thus in the case of a spherical quadrilateral the *s. m. c.* M is the intersection of the connectors of the middle points of opposite sides, and is such that

$$\text{arc } MA+\text{arc } MB=-(\text{arc } MC+\text{arc } MD).$$



For, if X is the *sum* of arc MA and arc MB,

$$\sin X=2 \sin MA'. \cos \tfrac{1}{2} AB, \text{ by § 4.}$$

Now, if P be the pole of B′D′,

$$\cos PA + \cos PB = 2 \cos PA'. \cos \tfrac{1}{2} AB.$$

and

$$\cos PA' = \sin MA'. \sin M.$$

$$\therefore \quad \cos PA + \cos PB = \sin x. \sin M.$$

In other words, $\sin X \sin M = \sin \alpha + \sin \beta$, where α, β are the perpendiculars from A, B on B′D′.

Similarly, the resultant of arcs MC, MD is Y, such that

$$\sin Y \sin M = \text{sum of sines of perpendiculars from C, D.}$$

Hence $\quad \sin X = -\sin Y. \quad \therefore X = -Y.$

Cor. 1. In a spherical quadrilateral, if A′B′C′D′ be the mid-points of AB, BC, CD, DA, and M be the *s. m. c.*

$$\sin MA'. \cos \tfrac{1}{2} AB = \sin MC' \cos \tfrac{1}{2} CD,$$
$$\sin MB'. \cos \tfrac{1}{2} BC = \sin MD' \cos \tfrac{1}{2} AD;$$

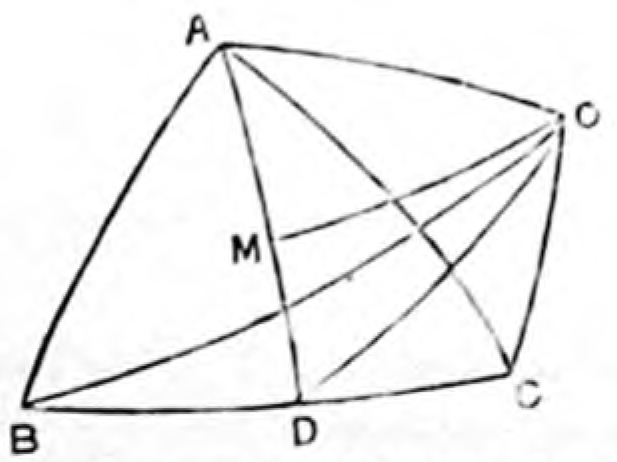
6. Generally, the resultant X of p. OA and q. OB may be defined as follows:

Divide, the arc AB at C into parts whose sines are inversely as $p : q$, then X is represented by λ.OC, such that

$$\sin X = \lambda. \sin OC = (p \cos AC + q \cos CB). \sin OC.$$

7. We can now proceed to give a geometrical construction for the resultant of arcs OA, OB, OC where O is any point whatever.

By § 4 the resultant of arcs OB, OC is $2 \cos \frac{1}{2} a \sin OD$, and the resultant of this and OA is λ OM, where M divides AD inversely as $1 : 2 \cos \frac{1}{2} a$ by § 6.



Hence, the resultant of arc OA, OB, OC, is X or $\lambda \cdot$OM such that

$$\sin X = \lambda \cdot \sin OM = \sin OM \ (2 \cos \tfrac{1}{2} a \cos MD + \cos MA)$$
$$= \sin OM. (\cos MA + \cos MB + \cos MC).$$

Similarly, the resultant of arcs OA, OB, OC, OD is X or λOM, such that

$$\sin X = \lambda \sin OM = \sin OM \ \Sigma \cos MA,$$

where M is the *s. m. c.* of ABCD; and so on.

8. Next, let M be a point defined by the relation

$$pMA + qMB + rMC = 0.$$

In this case it is readily seen that the point M is the point of concourse of the arcs AD, BE, CF, where, D, E, F divide the sides of ABC into parts whose sines are in the inverse ratios $p:q:r$.

This point M is the *spherical mean centre* of ABC for *multiples* p, q, r.

If O is any other point the resultant of p OA, q OB, r OC is seen to be X or λ OM, such that

$$\sin X = \lambda \sin OM = \sin OM. \Sigma(p \cos MA);$$

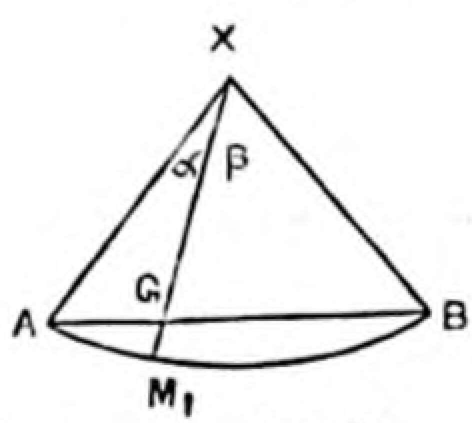
and so on.

9. The *s.m.c.* of a set of points may be obtained from statical considerations as follows:

Let G denote the centroid of masses p, q, r.........placed at A, B, C, ...then the *spherical mean centre* of ABC......... ; for multiples p, q, r......coincides with the projection of G on the spherical surface.

For, the centroid G, of p, q, divides the chord AB into parts inversely as $p:q$; so that its projection divides the arc AB into parts α, β connected by the relation

$$\sin \alpha : \sin \beta = \triangle AXG_1 : \triangle BXG_1$$
$$= AG_1 : BG_1 = q : p.$$



Hence the result is true in the case of A, B.

Similarly, its truth is established in the case of any number of points A, B, C,.........

10. The *s. m. c.* of a set of $(x+y)$ points lies on the arc joining the *s. m. c.* of the first x points to that of the remaining y points.

The proof is obvious statically. Hence, we infer that the several arcs joining the *s. m. c.* of any x points to that of the remaining y points are concurrent.

Cor. 1. The arcs joining the mid-points of opposite connectors of four points are concurrent.

Cor. 2. If in a $\triangle$ ABC D, E, F be the mid-points of the sides and L, M, N those of the arcs joining the ortho-centre to A, B, C, then DL, EM, FN are concurrent; and the point of concurrence lies on the arc joining the ortho-centre to the median centre.

11. If $x_1, x_2, x_3, \ldots\ldots$ denote the sines of the perpendicular arcs from a set of points $A_1, A_2, A_3, \ldots\ldots$ on any arc X, and x the sine of the perpendicular from M their *s. m. c.* for multiples $m_1, m_2, m_3, \ldots\ldots\ldots\ldots$ then $m.x = \Sigma(m_1\, x_1)$, where m denotes $\Sigma(m_1 \cos MA_1)$

This follows from the statical property of the *centroid* of $m_1, m_2, \ldots$ placed at $A_1, A_2, \ldots\ldots\ldots$

For, the *actual* distances of $m_1, m_2, \ldots\ldots$ from the plane of the arc X are proportional to $x_1, x_2, \ldots\ldots$ so that the *actual* distance of the *centroid* from X is $R.\Sigma(m_1\, x_1)/\Sigma(m_1)$. Also, the distance of the centroid from the centre of the sphere is d such that

$$d.\Sigma(m_1) = \Sigma(m_1\, d_1),$$

where $d_1, d_2, \ldots\ldots$ are the actual distances of $A_1, A_2, \ldots\ldots$ from the plane of the polar circle of M; and $d_1, d_2, \ldots\ldots\ldots$ are equal to $R \cos MA_1$, $R \cos MA_2 \ldots\ldots\ldots$

Thus $$d\Sigma m_1 = R.\Sigma_1\,(m_1 \cos MA_1) = m.R.$$

But $$x = \frac{R.\Sigma(m_1\, x_1)}{d\Sigma(m_1)}.$$

Hence $$m.x = \Sigma(m_1\, x_1).$$

The result may also be deduced trigonometrically from Stewart's Theorem.

Cor. 1. If X is an arc passing through M,

$$\Sigma(m_1\, x_1) = 0.$$

Thus the *s. m. c.* may also be defined as the point determined by the above equation.

Cor. 2. The sine of the perpendicular from M on any side $A_r\, A_{r+1}$ of $A_1\, A_2 \ldots\ldots$, varies as the sum of the sines of the perpendiculars from the remaining vertices on it.

Cor. 3. If O be the pole of X

$$m.\cos OM = \Sigma(m_1 \cos OA_1),$$

since $x. x_1, x_2 \ldots\ldots$ are equal to $\cos OM$, $\cos OA_1$, $\cos OA_2 \ldots\ldots\ldots$

Cor. 4. If D, E, F be the middle points of the sides of the triangle ABC, the median centre M of ABC is the *s. m. c.* of DEF for multiples $\cos \frac{1}{2} a$, $\cos \frac{1}{2} b$, $\cos \frac{1}{2} c$.

For, we have $\cos OB + \cos OC = 2 \cos \frac{1}{2} a \cos OD$, and two similar results.

$\therefore$ $$\Sigma(\cos OA) = \Sigma(\cos \tfrac{1}{2} a \cos OD).$$

But $$\Sigma \cos OA \propto \cos OM.$$

Thus $\Sigma \cos \frac{1}{2} a \cos OD \propto \cos OM$, and therefore M is the *s m.c.* of D, F, F for multiples $\cos \frac{1}{2} a$, $\cos \frac{1}{2} b$, $\cos \frac{1}{2} c$.

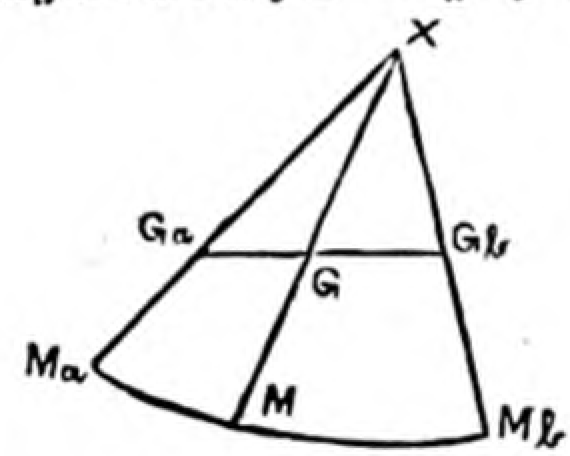
Similarly, if A, B, C, D...be a set of points and X, Y, Z...be the mid-points of AB, BC, CD,..., then the *s. m. c.* of X, Y, Z...for multiples $\cos\frac{1}{2}$ AB, $\cos\frac{1}{2}$ BC, $\cos\frac{1}{2}$ CD......coincides with that of A, B, C, D...for equimultiples.

12. If M_a is the *s. m. c.* of a set of x points A_1, A_2......and M_b that of another set of y points B_1, B_2.........., the *s. m. c.* of the combined set divides the arc $M_a M_b$ in the inverse ratio $\Sigma(\cos M_a A_1) : \Sigma(\cos M_b B_1)$.

For, if d_a, d_b, denote the distances of the centroids G_a, G_b of the two sets from the centre of the sphere, the certroid G of the combined set divides $G_a G_b$ in the inverse ratio $x : y$. Also

$$xd_a : y\,d_b = \Sigma(\cos M_a A_1) : \Sigma(\cos M_b B_1), \text{ by } \S\,\mathbf{11};$$

and $$\sin M_a M : \sin M_b M = xd_a : ydb.$$



Hence

$$\sin M_a M : \sin M_b M = \Sigma(\cos M_a A_1) : \Sigma(\cos M_b B_1).$$

Cor. 5. If two triangles ABC, A′B′C′ be such that $\cos a + \cos b + \cos c = \cos a' + \cos b' + \cos c'$, then the *s.m.c.* of the six points ABC A′B′C′ bisects the join of the median centres of the triangles.

For, in this case $\Sigma\sin^2\frac{1}{2}a = \Sigma\sin^2\frac{1}{2}a'$, and therefore the sum of the squares of the sides of the *chordal* triangles are equal, proving that the centroids of these triangles are equidistant from the centre of the sphere. Hence the result.

13. Let M be the *s. m. c.* of a triangle ABC for multiples n_1, n_2, n_3 and let the radius to M meet the plane of the chordal triangle at G. Then, if x, y, z be the areal coordinates of G, we have

$$x : y : z = n_1 : n_2 : n_3.$$

For, we know that n_1, n_2, n_3 are proportional to the triangular coordinates of M (Casey: § 76). Also x, y, z are proportional to the volumes of the tetrahedra standing on BGC, CGA, AGB and having for their common vertex the centre of the sphere; and these are as $n_1 : n_2 : n_3$. Hence the result.

Similarly if A, B, C, D...be any number of points on a small circle and M be their *s. m. c.* for multiples n_1, n_2, n_3,......the corresponding point

G on the plane of the small circle will have areal coordinates proportional to the Staudtians of the triangles AMB, BMC.........

Cor. 1. The in centre of the chordal triangle projects into the point whose triangular coordinates are proportional to the chords a_1, b_1 c_1, of the chordal triangle. But $a_1 : b_1 : c_1 = \sin \frac{1}{2} a : \sin \frac{1}{2} b : \sin \frac{1}{2} c$. Hence the projection is the point $\sin \frac{1}{2} a : \sin \frac{1}{2} b : \sin \frac{1}{2} c$.

Cor. 2. The orthocentre of the chordal triangle projects into the point whose triangular coordinates are as $\tan A_1 : \tan B_1 : \tan C_1$. But A_1=half the angle subtended by BC at the pole of the circumcircle (Casey : XXXV, 1).

$$\therefore \qquad \tan A_1 \propto \tan \tfrac{1}{2} a \operatorname{cosec} (S-A) \propto \cot (S-A).$$

Hence, the projection is the point $\cot S-A : \cot S-B : \cot S-C$.

14. We have

$$\Sigma(m_1 \cos OA_1) = m \cos OM \text{ by §11, } Cor.\ 3.$$

Let O coincide with A_1A_2......in succession; then

$$m_1 + m_2 \cos A_1A_2 + m_3 \cos A_1A_3 + \ldots\ldots = m \cos A_1M$$
$$m_1 \cos A_2A_1 + m_2 + m_3 \cos A_2A_3 + \ldots\ldots = m \cos A_2M$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

Multiply these equations by m_1, m_2, m_3...............and add :

$$\Sigma(m_1^2) + 2\Sigma(m_1m_2 \cos A_1A_2) = m\Sigma(m_1 \cos A_1N) = m^2, \text{ by §11.}$$

In particular, for a triangle ABC, if n_1, n_2, n_3 be the triangular co-ordinates of any point M,

$$\Sigma n_1^2 + 2\Sigma(n_1n_2 \cos c) = n^2.$$

15. For any two sets of points $A_1A_2 \ldots\ldots\ldots A_x$; $B_1B_2 \ldots\ldots\ldots B_y$ and multiples $m_1, m_2 \ldots\ldots m_x$; $n_1, n_2 \ldots\ldots n_y$, to prove that

(i) $\Sigma\{m_p n_q \cos A_p B_q\} = mn \cos M_a M_b$;

(ii) $\Sigma(m_p n_q \cos OA_p . \cos OB_q) = m\, n \cos OM_a . \cos OM_b$.

We have

$$\left.\begin{array}{l} \Sigma m_p \cos B_1A_p = m \cos B_1 M_a \\ \Sigma m_p \cos B_2A_p = m \cos B_2 M_a, \\ \ldots \quad \ldots \quad \ldots \\ \ldots \quad \ldots \quad \ldots \end{array}\right\} [p=1, 2, \ldots\ldots x].$$

Multiply these by n_1, n_2......and add ; the result is

$$\Sigma(m_p n_q \cos A_p B_q) = m\Sigma n_q \cos B_q M_a, \quad [q=1, 2, 3 \ldots\ldots y]$$
$$= m\, n \cos M_aM_b.$$

Again

$$\Sigma m_p \cos OA_p = m \cos OM_a. \quad [p.=1, 2 \ldots\ldots x]$$

and $$\Sigma n_q \cos OB_q = n \cos OM_b. \quad [q=1, 2, \ldots\ldots y]$$

Multiplying we get

$$\Sigma(m_p\, n_q \cos OA_p{\cdot}OB_q)=m.n.\ \cos OM_a \cos OM_b.$$

Cor. 1. The two sets have the same *s. m. c.*, if $M_a\, M_b=0$, the condition for which is

$$\Sigma(m_p\, n_q \cos A_p\, B_q)=m.n.$$

Cor. 2. If the arc $M_a\, M_b$ is a quadrant, $\cos M_a\, M_b=0$, and, therefore the condition for this is

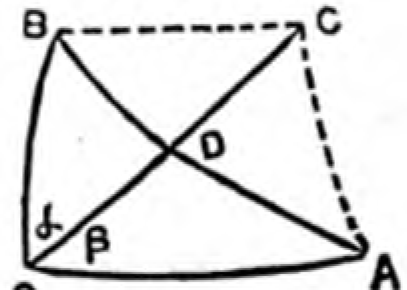
$$\Sigma(m_p\, n_q \cos A_p\, B_q)=0.$$

Cor. 3. Two spherical triangles ABC, A′B′C′ have a common median centre, if the sum of the cosines of all the arcs joining their vertices is equal to

$$\sqrt{\{3+2(\cos a+\cos b+\cos c)\}\ \{3+2(\cos a'+\cos b'+\cos c')\}}$$

16. Consider the resultant of the set $\Sigma m_p n_q\, A_pB_q$ for different values of p & q. By § 8, the resultant of $n_1A_1B_1$, $n_2A_1B_2$.........$n_yA_1B_y$ is represented by $n\ A_1M_b$, where $n=\Sigma(n_y \cos M_bB_q)$; that of $n_1A_2B_1$, $n_2A_2B_b$.........$n_yA_2B_y$ by $n\ A_2M_b$; &c. Also, the resultant of $m_1A_1M_b$, $m_2A_2M_b$.........is represented by $m{\cdot}M_aM_b$ where $m=\Sigma(m_p \cos M_aA_p)$. Hence the resultant of the set $\Sigma m_pn_qA_pB_q$ is $mn\ M_aM_b$.

17. Returning to § 4 let us examine in detail how far the analogy between *plane vectors* and *spherical arcs* can be stretched.



Let OA, OB be two great circle arcs; then their *resultant* is obtained as follows:

Through A, B draw parallels (arcs of small circles) to OB, OA respectively intersecting at C; OC shall be the resultant required. For, denoting sin OA, sin OB, sin OC by x, y, z

$$z \sin \beta=y \sin(\alpha+\beta)\,;\ z \sin \alpha=x \sin(\alpha+\beta).$$

$$\therefore\qquad x : y : z=\sin \alpha : \sin \beta : \sin(\alpha+\beta).$$

Now $x \sin \beta=y \sin \alpha$ is the condition that D should the middle point of AB; and

$$\begin{aligned} z=x \sin(\alpha+\beta)/\sin \alpha &=\sin AB{\cdot}\sin B/\sin \alpha\\ &=\sin AB{\cdot}\sin OD/\sin BD\\ &=2 \cos \tfrac{1}{2} AB{\cdot}\sin OD,\end{aligned}$$

so that OC represents the resultant of OA, OB.

Cor. 1. Put $\theta=\alpha+\beta$, then

$$z \sin \beta = y \sin \theta;\ z \sin \alpha = x \sin \theta.$$

Multiply by $\cos \alpha$, $\cos \beta$ and add :

$$z = x \cos \beta + y \cos \alpha.$$

That is z is the sum of the resolved parts of x and y along the resultant.

Cor. 2. Again $z \cos \beta = x \cos^2 \beta + y \cos \alpha \cos \beta$

$$= x - x \sin^2 \beta + y \cos \alpha \cos \beta.$$

$$= x - y \sin \alpha \sin \beta + y \cos \alpha \cos \beta = x + y \cos \theta.$$

$$\therefore \quad z^2 = x^2 + y^2 + 2\,xy \cos \theta.$$

Cor 3. The sum of the resolved parts of x and y along any arc through O is equal to the resolved part of z along the same.

For
$$z \sin \beta = y \sin \theta,$$
$$z \cos \beta = x + y \cos \theta.$$

$$\therefore \quad z \cos (\beta+\lambda) = z \cos \beta \cos \lambda - z \sin \beta \sin \lambda$$
$$= x \cos \lambda + y (\cos \theta \cos \lambda - \sin \theta \sin \lambda),$$
$$= x \cos \lambda + y \cos (\theta+\lambda).$$

18. The resultant of any number of arcs OA, OB, OC,...is found as in the case of plane vectors by resolving along two perpendicular arcs through O and compounding the components.

Thus if X, Y, denote the sums of components of x, y,...along these rectangular arcs the sine of the resultant is Z, such that

$$Z^2 = X^2 + Y^2.$$

and the inclination of the resultant is $\tan^{-1} (Y/X)$

Cor 1. The conditions that OA, OB,...should have a *null* or *zero* resultant are that $X=0$, and $Y=0$; and conversely, if these conditions are satisfied the resultant is zero; that is, the point O is the *s. m. c.* of the system of points A, B, C,...

Cor 2. Defining the *moment* of a spherical arc round any point as the product of the sine of the arc into the sine of the perpendicular from the point on the arc, it is easy to see that the moment of the resultant Z is the sum of the moments of the components. Hence follow other conditions for zero resultant, as in Statics.

19. If we replace a set of points A_1, A_2.........by a set of arcs, we have a definite arc called *the spherical mean axis* corresponding to the *s. m. c.* of the points A_1, A_2, whose properties follow from the reciprocal therems relating to poles and polars.

Equi-Brocardian Triangles.

By J. C. Swaminarayan, M.A.

1. Let BC be a given base. On both sides of BC, describe equilateral triangles BPC and BQC. If A is the vertex of any triangle on the base BC such that its Brocard angle is equal to a known angle w, we have from $\triangle$ APB

$$\begin{aligned} AP^2 &= a^2+c^2-2ac\cos(B+60^\circ) \\ &= a^2+c^2-ac\cos B+2ac\sin B\sin 60^\circ \\ &= a^2+c^2-\tfrac{1}{2}(a^2+c^2-b^2)+2\triangle\sqrt{3} \\ &= \tfrac{1}{2}(a^2+b^2+c^2)+2\triangle\sqrt{3} \\ &= 2\triangle(\cot w+\sqrt{3}) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1) \end{aligned}$$

Similarly, from $\triangle$AQB

$$AQ^2=2\triangle(\cot w-\sqrt{3}) \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

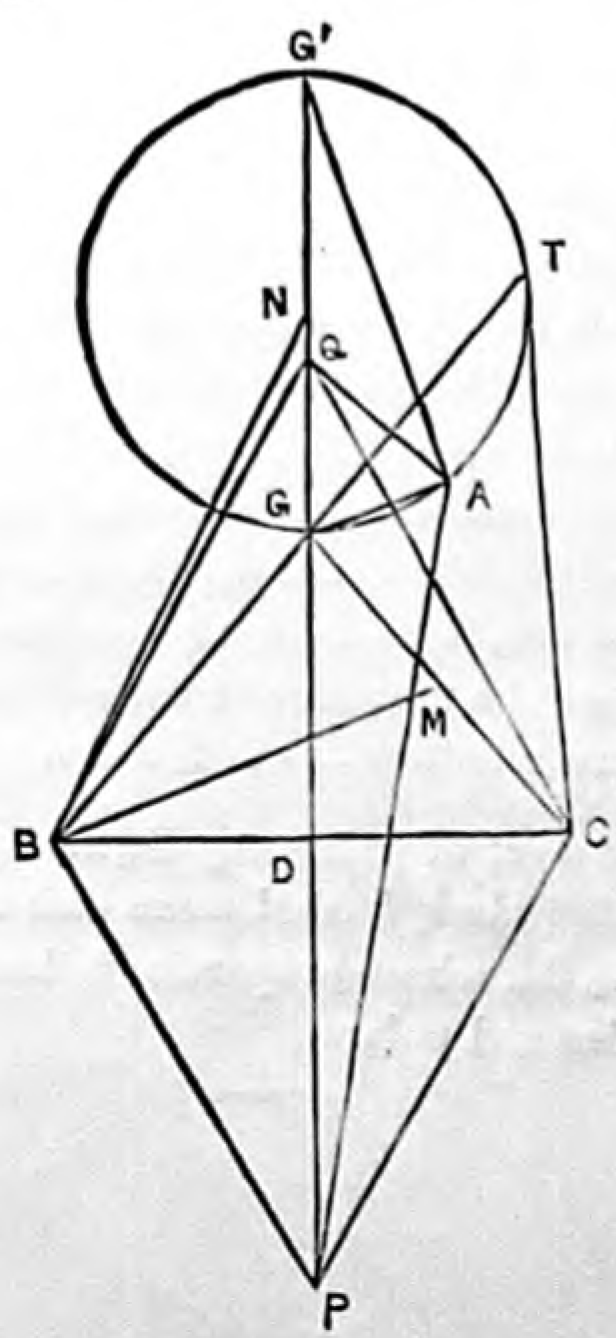
The latter result shows that $\cot w \not< \sqrt{3}$.

2. From this it is apparent that the ratio AP : AQ depends only on the value of the Brocard angle. Because

$$\frac{AP}{AQ}=\left\{\frac{\cot w+\sqrt{3}}{\cot w-\sqrt{3}}\right\}^{\frac{1}{2}}.$$

Hence the locus of the vertex A of the triangle whose Brocard angle is equal to w can be obtained by bisecting $\angle$PAQ internally and externally by lines AG and AG′, cutting PQ in G,G′ respectively. The locus of A will be the circle on GG′ as diameter.

3. Putting, AP$=\alpha$, AQ$=\beta$, we can easily prove that if D is the middle point of BC,



$$DG = PG - PD = \frac{a\sqrt{3}}{2}\left\{\frac{2\alpha}{\alpha+\beta} - 1\right\} = \frac{a\sqrt{3}}{2}.\frac{\alpha-\beta}{\alpha+\beta}$$

Similarly $$DG' = \frac{a\sqrt{3}}{2}.\frac{\alpha+\beta}{\alpha-\beta}.$$

Hence, if N is the centre of the circle on GG′ as diameter,

$$\begin{aligned}
DN &= \tfrac{1}{2}(DG + DG') \\
&= \frac{a\sqrt{3}}{4}\left\{\frac{\alpha+\beta}{\alpha-\beta} + \frac{\alpha-\beta}{\alpha+\beta}\right\} \\
&= \frac{a\sqrt{3}}{4}\left\{\frac{2(\alpha^2+\beta^2)}{\alpha^2-\beta^2}\right\} \\
&= \frac{a\sqrt{3}}{2}\left\{\frac{AP^2+AQ^2}{AP^2-AQ^2}\right\} \\
&= \frac{a\sqrt{3}}{2}\left\{\frac{\cot w+\sqrt{3}+\cot w-\sqrt{3}}{\cot w+\sqrt{3}-\cot w+\sqrt{3}}\right\} \\
&= \frac{a}{2}\sqrt{3}.\frac{\cot w}{\sqrt{3}} = \frac{a}{2}\cot w
\end{aligned}$$

4. Now

$$\cot B\hat{N}D = \frac{DN}{BD} = \frac{\frac{1}{2}a \cot w}{\frac{1}{2}a} = \cot w.$$

$\therefore$ $$B\hat{N}D = w.$$

Hence BC subtends at N an angle equal to $2w$.

5. Taking BC and DN as the axes of x and y, we obtain the equation of the locus of A in the form

$$\begin{aligned}
x^2 + (y - \tfrac{1}{2}a \cot w)^2 &= GN^2 \\
&= \{\tfrac{1}{2}a \cot w - DG\}^2
\end{aligned}$$

But
$$\begin{aligned}
DG &= a\,\frac{\sqrt{3}}{2}\,\frac{\alpha-\beta}{\alpha+\beta} \\
&= a\frac{\sqrt{3}}{2}.\frac{\sqrt{\cot w+\cot 30^\circ}-\sqrt{\cot w-\cot 30^\circ}}{\sqrt{\cot w+\cot 30^\circ}+\sqrt{\cot w-\cot 30^\circ}} \\
&= \frac{a}{2}\cot w + \frac{a}{2}\sqrt{\cot^2 w - 3}.
\end{aligned}$$

$\therefore$ $$x^2 + (y - \tfrac{1}{2}a \cot w)^2 = \tfrac{1}{4}a^2 \cot^2 w - \tfrac{3}{4}a^2.$$

$\therefore$ $$x^2 + y^2 - ay \cot w + \tfrac{3}{4}a^2 = 0. \qquad \ldots \qquad \ldots \quad (3)$$

The circle (3) is called Neuberg's circle, after the distinguished geometer who first studied its properties.

6. It is interesting to notice that

$$\angle NBD = 90^\circ - w,$$
$$\angle NBQ = 30^\circ - w,$$
and $$\angle NBP = 150^\circ - w.$$

$$\therefore \qquad \frac{NQ}{NP}=\frac{\Delta NBQ}{\Delta NBP}=\frac{\frac{1}{2}NB\cdot BQ\sin(30^\circ-\omega)}{\frac{1}{2}NB\cdot AP\sin(150^\circ-\omega)}$$

$$=\frac{\sin 30^\circ \sin\omega(\cot w-\cot 30^\circ)}{\sin 150^\circ \sin w(\cot w-\cot 150^\circ)}$$

$$=\frac{\cot w-\cot 30^\circ}{\cot w+\cot 30^\circ}=\frac{AQ^2}{AP^2}.$$

Also $\qquad NQ\cdot NP=\frac{1}{2}a(\cot w-\cot 30^\circ).\frac{1}{2}a(\cot w+\cot 30^\circ)$

$=\frac{1}{4}a^2(\cot^2 w-3)$

$=$square of the radius of Neuberg's circle from (3),

$=NA^2$.

Hence NA touches the circum-circle of ΔAPQ and $\angle NAQ=\angle QPA$.

It is obvious that Neuberg's circle cuts the circum-circle of ΔQAP orthogonally.

7. As G and G′ lies on Neuberg's circle, the Brocard angles of the triangles BGC and BG′C will each be equal to w. In the triangle BG′C, draw BM the median through B. Then

$$\cot G\hat{B}M=\frac{2BG^2+2BM^2-\frac{1}{2}GC^2}{4\Delta GBC}$$

$$=\frac{2BG^2+BG^2+BC^2-\frac{1}{2}GC^2-\frac{1}{2}GC^2}{4\Delta BGC}$$

$$=\frac{BG^2+CG^2+BC^2}{4\Delta BGC}, \text{ since } BG=CG,$$

$$=\cot w.$$

$$\therefore \qquad GBM=w.$$

If $\angle BGC$ is denoted by 2λ, then

$$\frac{GB}{GM}=\frac{2}{1}=\frac{\sin(2\lambda+w)}{\sin w}$$

$$\therefore \qquad \sin(2\lambda+w)=2\sin w \qquad \dots \qquad \dots \qquad (4)$$

Similarly, if $\angle BG'C=2\lambda'$,

$$\sin(2\lambda'+w)=2\sin w.$$

Thus $\angle BGC$ and $\angle BG'C$ are determined by the equation

$$\sin(2V+w)=2\sin w,$$

the greater value of V corresponding to $\angle BGC$ and the smaller to $\angle BG'C$.

In fact, BGC and BG′C are the vertical angles of two isosceles triangles on the base BC, having their Brocard angles equal to w. It is a well-known fact that Steiner-angles satisfy the equation

$$\sin(2V+w)=2\sin w.$$

Hence the Steiner angles may be defined as the vertical angles of two isosceles triangles on a given base such that their Brocard angles are equal to a given angle less than 30°.

8. Casey, defines Steiner angles as the angles which the tangents from B or C to Neuberg's circle make with BC. We can easily prove that the angles which these tangents make with BC are equal to the angles BGC and BG′C. Before doing so, we must notice the important fact that the length of the tangent from B or C to Neuberg's circle (3) is equal to $\sqrt{\frac{a^2}{4}+\frac{3a^2}{4}}$ or a.

Let BG be produced to cut Neuberg's circle again in T; then

$$BG \cdot BT = BC^2,$$

and $$\angle GBC = \angle GCB = \angle BTB.$$

$$\therefore \quad BC = CT.$$

Hence CT is a tangent to Neuberg's circle and

$$\triangle BGC \text{ is similar to } \triangle BCT.$$

$$\therefore \quad \angle BGC = \angle BCT.$$

Similarly, if CT′ is the other tangent from C, BT′ passes through G′, and $\angle BG'C = \angle BCT'$.

Thus our definition of Steiner angles agrees with Casey's definition.

SHORT NOTES.

The General Conic in Trilinears.

1. *The discrimination of the conics given by the general equation.*

Let $u\alpha^2+v\beta^2+w\gamma^2+2u'\beta\gamma+2v'\gamma\alpha+2w'\alpha\beta=0$ be the conic, which will be denoted by $(u, v, w, u', v', w')(\alpha, \beta, \gamma)^2=0$.

By multiplication of determinants we easily prove that when a conic is transformed by any linear substitution

$$X=l_1x+m_1y+n_1z$$
$$Y=l_2x+m_2y+n_2z$$
$$Z=l_3x+m_3y+n_3z$$

the discriminant of the transformed equation is equal to the discriminant of the original equation multiplied by the square of the determinant formed by the l, m, n's. Askwith: *Analytical Geometry*, § 353; Bromwich, *Quadratic Forms*. We have here

$$x=\alpha x_1+\beta x_2+\gamma x_3,$$
$$y=\alpha y_1+\beta y_2+\gamma y_3,$$
$$z=1=\alpha+\beta+\gamma;$$

where (x, y) are the rectangular coordinates of the point (α,β,γ); $(x_1y_1)\ldots$ are the vertices of the triangle of reference, and the determinant of the substitution is 2Δ.

If, therefore $(abcfgh)(x,y,1)^2$ transforms to $(u, v, w, u', v', w')(\alpha\beta\gamma)^2$, we have

$$\begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix} = 4\Delta^2 \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$$

Also $\qquad (a, b, c, f, g, h)(x, y, 1)^2+\lambda$

transforms to $\qquad (u, v, w, u', v', w')(\alpha, \beta, \gamma)^2+\lambda(\alpha+\beta+\gamma)^2$,

or $\qquad (u+\lambda, v+\lambda, w+\lambda, u'+\lambda, v'+\lambda, w'+\lambda)(\alpha, \beta, \gamma)^2$

Hence
$$4\Delta^2\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c+\lambda \end{vmatrix} = \begin{vmatrix} u+\lambda, & w'+\lambda, & v'+\lambda \\ w'+\lambda, & v+\lambda, & u'+\lambda \\ v'+\lambda, & u'+\lambda, & w+\lambda \end{vmatrix} = \begin{vmatrix} u, & w', & v', & 1 \\ w', & v, & u', & 1 \\ v', & u', & w, & 1 \\ -\lambda, & -\lambda, & -\lambda, & 1 \end{vmatrix}$$

Equating co-efficients of λ

$$4\Delta^2\ (ab-h)^2=-\begin{vmatrix} u, & w', & v', & 1 \\ w', & v, & u', & 1 \\ v', & u', & w, & 1 \\ 1, & 1_1 & 1, & 0 \end{vmatrix}=-\sigma, \text{ say.}$$

Thus the conic is an ellipse, parabola, or hyperbola according as σ is negative, zero or positive.

2. *To obtain the axes of the general conic.*

When referred to its centre $(a,b,c,f,g,h)\ (x,y,1)^2=0$ becomes

$$ax^2+2\ hxy+by^2+\frac{D}{ab-h^2}=0, \text{ where } D=\begin{vmatrix} a, h, g \\ h, b, f \\ g, f, c \end{vmatrix}$$

and the magnitudes of the axes are given by

$$\left(\frac{1}{r^2}+\frac{a}{\lambda}\right)\left(\frac{1}{r^2}+\frac{b}{\lambda}\right)=\frac{h^2}{\lambda^2}, \text{ where } \lambda=\frac{D}{b-h^2}.$$

We have already found D and $ab-h^2$ and it remains to find $a+b$.

Now $(a,b,c,f,g,h)(x,y,1)^2+k(x^2+y^2)$ transforms to

$$(u,v,w,u',v',w')(\alpha,\beta,)^2+k(x\gamma_1,^2+y\gamma_1,^2,\ldots,\ldots,x_2x_3+y_2y_3,\ldots,\ldots)(\alpha\beta\gamma)^2.$$

Therefore from § 1,

$$4\Delta^2\begin{vmatrix} a+k, & h \\ h, & b+k \end{vmatrix}=-\begin{vmatrix} u+ku_1, & w'+kn_1', & z'+kv_1', & 1 \\ w'+kw_1', & v+kv_1, & u'+ku_1', & 1 \\ v'+kv_1', & u'+ku_1', & w+kw_1, & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$$

where $u_1,\ldots\ldots u_1'$, are written for $x_1^2+y_1^2,\ \ldots,x_2x_3+y_2y_3,\ldots$

Now

$$-\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} \begin{aligned} &=U+V+W+2U'+2V'+2W' \\ &=(uv+uw+vw-u'^2-v'^2-w'^2-2uu'-2vv' \\ &\qquad -2ww'+2u'v'+2u'w'+2v'w') \\ &=f(u,u') \text{ say.} \end{aligned}$$

Equating the co-efficients of k on both sides

$$\begin{aligned} 4\Delta^2(a+b)&=\Sigma u\frac{\partial f(u_1,u_1')}{\partial u_1}+\Sigma u'\frac{\partial f(u_1,u'_1\theta)}{\partial u_1'} \\ &=\Sigma u(v_1+w_1-2u_1')+\Sigma u'(-2u_1'-2u_1+2v_1'+2w_1') \\ &=\Sigma u(x_2^2+y_2^2+x_3^2+y_3^2-2(x_2x_3+y_2y_3)) \\ &\qquad -2\Sigma u'(x_2x_3+y_2y_3+x_1^2+y_1^2-x_1x_3-y_1y_3-x_1x_2-y_1y_2) \\ &=\Sigma ua^2-2\Sigma u'bc\cos A, \end{aligned}$$

since $\qquad a^2=(x_2-x_3)^2+(y_2-y_3)^2$, and

$2bc\cos A=(x_3-x_1)^2+(y_3-y_1)^2+(x_1-x_2)^2+(y^2-y_2)_3-)x_2-x_3)^2-y)_2-y_3)^2$

$\qquad\qquad$ = co-efficient of u' written above.

This of course is the condition for a rectangular hyperbola.

The equation giving the axes is

$$\frac{\lambda^2}{r^4}+(a+b)\frac{\lambda}{r^2}+ab-h^2=0.$$

Writing $\rho=\dfrac{\lambda}{r^2}=\dfrac{D}{ab-h^2}\cdot\dfrac{1}{r^2}=-\begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix}\div r^2\begin{vmatrix} u, & w,' & v', & 1 \\ w', & v, & u', & 1 \\ v', & u', & w, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix}$

the axes are given by

$$4\,\Delta^2\rho^2+\rho(\Sigma ua^2-2\Sigma u'bc\cos A)-\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}=0.$$

3. *The conditions for a circle.*

The circumcircle being assumed to be $a^2\beta\gamma+b^2\gamma\alpha+c^2\alpha\beta=0$, for a circle, the expression

$$(u, v, w, u', v', w')\ (\alpha\beta\gamma)^2+\lambda\,\{\,a^2\beta\gamma-b^2\gamma(\beta+\gamma)-c^2\beta(\beta+\gamma)\,\}$$

must be divisible by $\alpha+\beta+\gamma$ and must therefore be identical with

$$(\alpha+\beta+\gamma)\,\{\,u\alpha+(v-\lambda c^2)\beta+(w-\lambda b^2)\gamma\,\}.$$

Equating co-efficients of $\beta\gamma$, we get

$$2u'-\lambda(b^2+c^2-a^2)=v+w-\lambda(b^2+c^2).$$

Hence $\qquad \lambda=\dfrac{v+w-2u'}{a^2}=\dfrac{u+w-2v'}{b^2}=\dfrac{u+v-2w'}{c^2}$,

by symmetrical considerations.

A. C. L. Wilkinson.

On Tangential Equations.

The following article is the development of a method indicated in R. A. Roberts' *Examples on Conics* (1884).

Let $a^2l^2+b^2m^2=1$ be the tangential equation of an ellipse referred to its axes and $(Xl+Ym-1)^2-R^2(l^2+m^2)=0$, that of a circle [centre (X,Y,) radius R]; then

$$a^2l^2+b^2m^2-1+\frac{\lambda^2}{l^2}\{\,(Xl+Ym-1)^2-R^2(l+m^2)\,\}=0, \qquad \ldots \quad (1)$$

denotes a cotangential conic. (*vide*: p. 30, Ex. 44)

When (1) splits up into factors, it represent two points. The discriminant is

$$\frac{X^2}{a^2-\lambda^2}+\frac{Y^2}{b^2-\lambda^2}+\frac{R^2}{\lambda^2}-1 \ =0.$$

For the values of λ derived from this equation the two points are the extremities of a diagonal of the tangential quadrilateral. At the same time, for these particular values of λ, the points as above determined are the intersections of

$$(a^2-\lambda)^2\, l^2+(b^2-\lambda^2)m^2-1=0, \text{ and } Xl+Ym-1=0\,;$$

i.e., of $$\frac{x^2}{a^2-\lambda^2}+\frac{y^2}{b^2-\lambda^2}-1 \ =0, \text{ and } \frac{xX}{a^2-\lambda^2}+\frac{yY}{b^2-\lambda^2}=1.$$

Hence, the opposite vertices of the cotangential quadrilateral of a circle and a conic lie on a confocal, being the intersections of the confocal with the polar of the centre of the circle with respect to that confocal. ... ... (2).

Hence, also, if λ_1, λ_2, λ_3 are the three roots of the equation $\frac{X^2}{a^2-\lambda^2}=\frac{Y^2}{b^2-\lambda^2}+\frac{R^2}{\lambda^2}=1$, we have, as in Lamb's *Hydrodynamics*, §. 112,

$$X^2=\frac{(a^2-\lambda_1^2)(a^2-\lambda_2^2)(a^2-\lambda_3^2)}{a^2(a^2-b^2)},\ \ Y^2=\frac{(b^2-\lambda_1^2)(b^2-\lambda_2^2)(b^2-\lambda_3^2)}{b^2(a^2-b^2)},$$

$$R^2=\frac{\lambda_1^2\lambda_2^2\lambda_3^2}{a^2b^2}, \text{ results worked out in Roberts (p. 31).}$$

Thus $$X=\frac{a_1a_2a_3}{ac},\quad Y=\frac{\sqrt{(c^2-a_1^2)(c^2-a_2^2)(c^2-a_3^2)}}{bc},$$

$$R=\frac{\sqrt{(a^2-a_1^2)(a^2-a_2^2)(a^2-a_3^2)}}{ab}, \qquad \text{... (3).}$$

where $a_1a_2a_3$ are the sami major axes of the three confocals through the three pairs of vertices.

For the incircle the confocals are hyperbolasand for the ex-circle two of the confocals are ellipses. Again, if s be the semi-perimeter of the triangle formed by three tangents to the in circle and the ellipse, we have $Rs^2=R_1R_2R_3$ where R_1, R_2, R_3, are the ex-radii and R the in-radius. On substituting for R_1, R_2, R_3 and R from (3) and remembering the distinction between *in-* and *ex*-circles as above referred to

$$s=\frac{\sqrt{-(a^2-\alpha_1^2)(a^2-\alpha_2^2)(a^2-\alpha_3^2)}}{ab},$$

α_1, α_2, α_3 being the semi-axes of the confocal ellipses through the three vertices. ... ... (4).

From (3) we see that, if two of the vertices of a triangle move along confocal hyperbolas, the incentre is given by

$$acx=a_1a_2a_3,\ b^2c^2y^2=(c^2-a_1^2)\ (c^2-a_2^2)\ (c^2-a_3^2)$$

whence, eliminating a_3, the incentre moves on the ellipse

$$\frac{a^2x^2}{a_1^2a_2^2}+\frac{b^2y^2}{(c^2-a_1^2)(c^2-a_2^2)}=1,$$

a result given, but not worked out, by Roberts (p. 34, Ex. 46).

Also, when two of the vertices of a triangle move along confocal ellipses, one of the ex-centres lies on a fixed coaxial conic, as seen above and the third corner moves on a confocal ellipse, so that, from (4), $2s$ or the perimeter of the triangle is constant. In the case of a triangle of maximum perimeter the sides touch a confocal, so that its perimeter is $(a^2-a_1^2)/a_1b_1$ from (4). The invariant relation is

$$\frac{a_1}{a}+\frac{b_1}{b}=1, \text{ or } \left(\frac{a_1-a}{a}\right)^2=\frac{b_1^2}{b^2}=\frac{a_1^2-c^2}{b^2}.$$

$$\therefore \quad -b^2(a_1^2+a^2-2a_1a)+a^2(a_1^2-c^2)=0.$$

$$\therefore \quad a_1^2c^2-2a_1ab^2-a^4=0.$$

$$\therefore \quad a_1=\frac{ab^2\pm\sqrt{a^2b^4+a^4c^2}}{c^2}=\frac{ab^2\pm a\sqrt{a^4-a^2b^2+b^4}}{c^2}$$

For obvious reasons we neglect the plus sign as giving a value greater than a, so that

$$a^2-a_1^2=\frac{a^2}{c^4}\{\,(a^2-b^2)^2-b^4-(a^4-a^2b^2+b^4)+2b^2\sqrt{a^4-a_2^2b^2+b^4}\,\}$$

$$=\frac{a^2b^2}{c^4}\{\,2\sqrt{a^2-a^2b^2+b^4}-(a^2+b^2)\,\}$$

$$=\frac{3a^2b^2}{(a^2+b^2)+2\sqrt{a^4-a^2b^2+b^4}}.$$

Thus remembering that $b_1^2=a_1^2-c^2$, we have for the semi-perimeter

$$s=\frac{\sqrt{3}\,\{\,a^2+b^2+\sqrt{a^4-a^2b^2+b^4}\,\}}{(a^2+b^2+2\sqrt{a^4-a^2b^2+b^4})^{\frac{1}{2}}}$$

a result given in Wolstenholme, Problem 1060. ... ... (5).

Again taking the circle of curvature at a point P and the conic, three of the sides of the quadrilateral are coincident along the tangent at P, and the other common tangent of this circle and the ellipse cuts it in a point Q, such that P and Q lie on the same con-focal through P. ... ... (6).

Further, if ABC be a triangle inscribed in a conic and circumscribed to a confocal, D the point of contact of the latter with BC and E the common point of BC and the fourth common tangent of the in circle of

ABC and the inner confocal, it follows from (2) that A and E lie on on the same confocal through A. But, by a well-known theorem, A and D also lie on the same confocal. Hence E is the point of intersection of BC with the confocal through A other than D. But D and the common point of the common tangents of the osculating circle at D and the inner conic also lie on the same confocal. It follows therefore that the fourth common tangent of the incircle and in-conic of ABC passing through E, touches the osculating circle at D. Hence the fourth common tangent of the incircle and the in-conic of ABC touches the osculating circle at D.

Similarly, for the points of contact with the other sides. Hence the proposition that, if a triangle be inscribed in a conic and circumscribed to a confocal, the osculating circle at the point of contact of the sides of the triangle with the latter are all touched by the fourth common tangent of the inconic and the incircle of the triangle—a proposition otherwise proved in Roberts (p. 113, Ex. 193) ... ... (7).

Further, if the in circle of a self-conjugate triangle, with reference to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, be $(x-\alpha)^2+(y-\beta)^2=\lambda^2$, forming the invariant condition $\Theta'=0$ for the circumscription of a self-conjugate triangle, we have

$$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-1-r^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)=0,$$

which may be written $$\frac{\alpha^2}{a_1^2-h^2}+\frac{\beta^2}{b_1^2-h^2}+\frac{r^2}{h^2}-1=0,$$

where $$a_1^2=\frac{a^4}{a^2+b^2},\ b_1^2=\frac{b^4}{a^2+b^2},\ h^2=-\frac{a^2b^2}{a^2+b^2}.$$

On comparison with results (3), (4), we see that the intersection of the polar of (α,β) with respect to $\frac{x^2}{a_1^2-h^2}+\frac{y^2}{b_1^2-h^2}=1$, or $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, with the same determine two points the tangents from which to $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2+b^2}$, are also tangents to $(x-\alpha)^2+(y-\beta)^2=r^2$.

Hence it follows that if (α,β) lie on the director circle of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, since it polar with reference to the same touches $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2+b^2}$ at the foot of the $\perp r$ from (α,β) on the polar, the in-

circle touches the polar at the same point as the confocal; and the envelope of the circle is the confocal

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2+b^2}$$ (Roberts, p. 5, Ex. 67).

A. N. RAGHAVACHAR.

The Face of the Sky for March and April 1914.

Sidereal time at 8 p.m.

	March.				*April.*		
D.	H.	M.	S.		H.	M.	S.
1	6	34	12	...	8	37	22
8	7	1	48	...	9	4	58
15	7	29	23	...	9	32	34
22	7	56	59	...	10	0	9
29	8	24	45	...	10	27	45

From this table the constellations visible during the evenings of March and April can be ascertained by a reference to the positions as given in a star-atlas.

The Sun

enters the vernal equinox on March 21 at 4·41 P.M.

Phases of the Moon.

	March.					*April.*			
	D.	H.	M.			D.	H.	M.	
First Quarter ...	5	10	33	A.M.	...	4	1	11	A.M.
Full Moon ...	12	9	48	„	...	10	6	58	P.M.
Last Quarter ...	19	1	9	„	...	17	1	22	„
New Moon ...	26	11	39	„	...	25	4	52	„

Eclipses.

The eclipses of February and March are invisible at Madras.

Planets.

Mercury which was in superior conjunction with the sun on January 22, attains its greatest (E) elongation on February 22, is stationary on

March 1, is in inferior conjunction with the sun on March 10, is again stationary on March 23. It attains its greatest (W) elongation on April 7. It is in conjunction with the moon on February 26, March 24, and April 23.

Venus is in superior conjunction with the sun on February 12, and is an evening star after that date. It is in conjunction with the moon on March 28.

Mars which was in opposition to the sun on January 5, is stationary on February 13. It is in conjunction with the moon on March 7 and April 4, and with Neptune on April 21.

Jupiter which was in conjunction with the sun on January 20, is a morning star. It is in conjunction with the moon on February 22 and on April 19 at 5-43 A.M. and with Uranus on March 4.

Saturn is in quadrature to the sun on March 3. It is in conjunction with the moon on March 5, April 1 at 6 P.M. and on April 29.

Uranus was in conjunction with the sun on January 29; and will be so with the moon on February 23, March 22, and on April 18 at 4-20 A.M.

Neptune which was in opposition to the sun on January 18, becomes stationary on April 6. It is in conjunction with the moon on February 9, and on March 8 at 1-36 A.M.

V. RAMESAM.

SOLUTIONS.

Question 366.

(M. KANNAN, B.A., L.T.) :—In a triangle ABC, PP′ is a diameter of the circumcircle, and the Simson lines of P, P′ interest at ω. Shew that the length of the perpendicular from ω on PP′ is $2R \cos \alpha \cos \beta \cos \gamma$, where α, β, γ are the inclinations of PP′ to the sides of the triangle.

Additional solution by 'Zero' and M. Satyanarayana.

Take P as the origin of polar coordinates and PP′ as the initial line. If the vectorial angles of A, B, C be x, y, z the Simson-line of P is

$$r \cos (\theta - x - y - z) = 2R \cos x \cos y \cos z.$$

Hence the coordinates of the foot of the perpendicular from P on the Simson-line are

$$2\, R \cos x \cos y \cos z,\ (x+y+z).$$

When referred to the centre as the origin, the coordinates are therefore

$$2R \cos x \cos y \cos z - R \cos (x+y+z),\ (x+y+z).$$

Similarly, the foot of the perpendicular from P′ on its Simson-line is

$$2\, R \sin x \sin y \sin z + R \sin (x+y+z), \left(x+y+z-\frac{\pi}{2}\right).$$

The two Simson-lines will thus intersect at a point ω whose distanc from the initial line is

$$\{ 2\, R \sin x \sin y \sin z + R \sin (x+y+z) \} \cos (x+y+z) -$$
$$\{ 2\, R \cos x \cos y \cos z - R \cos (x+y+z) \} \sin (x+y+z).$$

Now $\quad \cos (x+y+z) = \cos x \cos y \cos z\, (1 - s_2)$

$\qquad \sin (x+y+z) = \cos x \cos y \cos z\, (s_1 - s_3)$

$\sin (x+y) \sin (y+z) \sin (z+x) = \cos^2 x \cos^2 y \cos^2 z\, (s_1 s_2 - s_3)$;

where s_1, s_2, s_3 denote the sums of the products of tangents 1, 2, 3 at a time.

Thus, the distance of ω from PP′ is

$$2\, R \cos^2 x \cos^2 y \cos^2 z \{ s_3 + s_1 - s_3)(1 - s_2) - (s_1 - s_3) \}$$
$$= -2\, R \cos^2 x \cos^2 y \cos^2 z\, (s_1 s_2 - s_3)$$
$$= -2\, R \sin (x+y) \sin (y+z) \sin (z+x)$$
$$= 2\, R \cos \alpha \cos \beta \cos \gamma,$$

since the inclinations of the sides of ABC to PP′ are $\left(\frac{\pi}{2} + \alpha + \beta\right)$, &c.

Question 467.

(N. Sankara Aiyar, M. A.) :—If O is the circumcentre of ABC and A′B′C′ the triangle whose sides bisect OA, OB, OC perpendicularly prove that AA′, BB′, CC′ are concurrent.

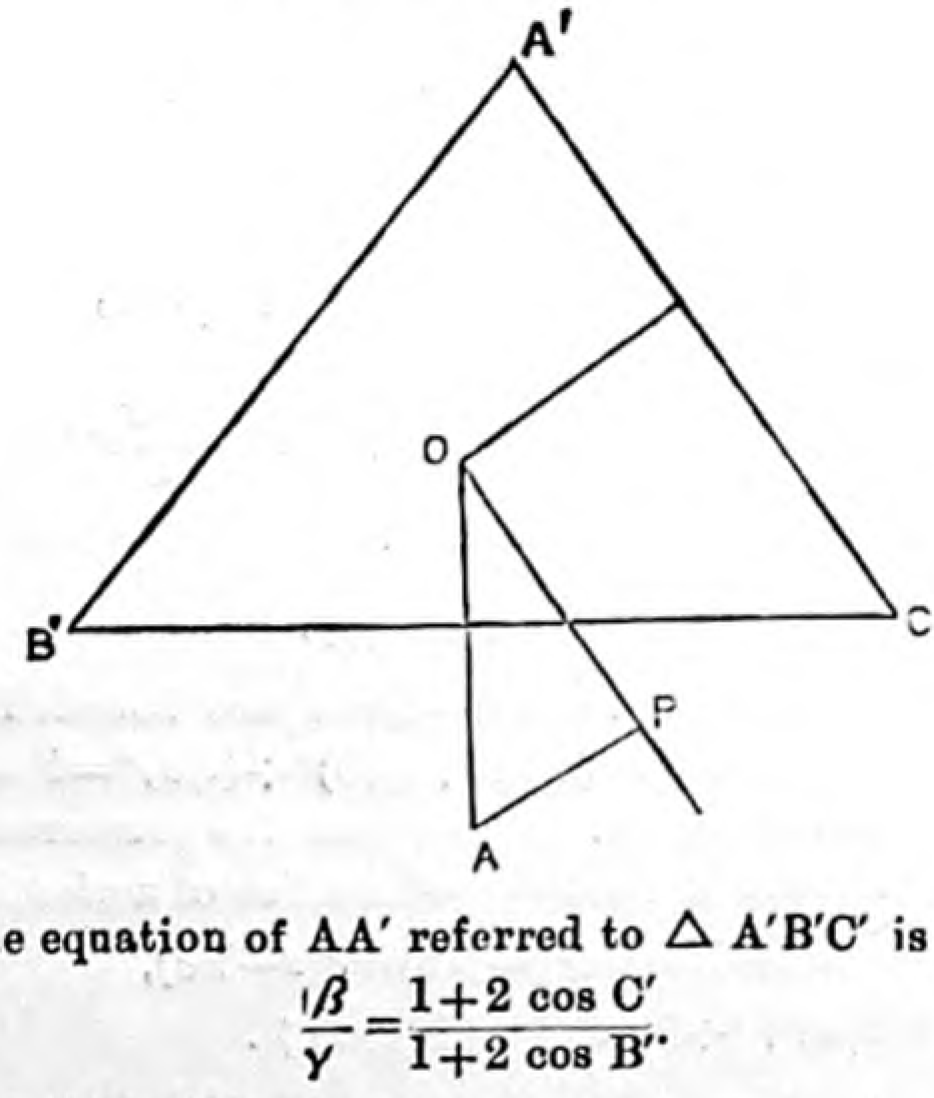
Additional Solution by Narasinga Rao, A.

It is evident that O is the incentre of A′B′C′.

Through O draw OP parallel to A′C′ and let AP be the ⊥ from A on this line.

Then the ⊥ from A on $A'C' = r + AP$ where r is the inradius of of A′B′C′.

But
$$r + AP = r\,(1 + 2\sin AOP)$$
$$= r\,(1 + 2\cos C').$$



Hence the equation of AA′ referred to △ A′B′C′ is
$$\frac{\beta}{\gamma} = \frac{1 + 2\cos C'}{1 + 2\cos B'}.$$

Thus AA′, BB′, CC′ are concurrent at
$$\left(\frac{1}{1 + 2\cos A'}, \frac{1}{1 + 2\cos B'}, \frac{1}{1 + 2\cos C'}\right)$$

Question 472.

(Selected) :—Evaluate
$$\int_0^\infty \frac{\cos x}{(1+x)^2}dx, \int_0^\infty \frac{\cos 2x}{(1+x^2)^2}dx$$

Additional Solution by N. Sankara Aiyar, M. A.

$$\text{Let } \mathrm{J}=\int_0^\infty \frac{\sin xy}{x\,(1+x^2)^2}\,dx$$

$$\therefore \qquad \frac{d^4\mathrm{J}}{dy^4}-2\,\frac{d^2\mathrm{J}}{dy^2}+\mathrm{J}=\int_0^\infty \frac{\sin xy}{x}\,dx=\frac{\pi}{2}.$$

$$\therefore \qquad \mathrm{J}=\mathrm{A}ye^{y}+\mathrm{B}ye^{-y}+\mathrm{C}e^{y}+\mathrm{D}e^{y}+\frac{\pi}{2}.$$

Now, J vanishes with $y=0$, and is infinite when $y=\propto$.

$$\therefore \qquad \mathrm{J}=\mathrm{B}ye^{-y}+\mathrm{D}e^{-y}+\frac{\pi}{2},\ \text{whence } \mathrm{D}=-\frac{\pi}{2}.$$

Differentiating

$$\frac{d\mathrm{J}}{dy}=\int_0^\infty \frac{\cos xy}{(1+x^2)^2}\,dx=-\mathrm{B}ye^{-y}+(\mathrm{B}-\mathrm{D})\,e^{-y}.$$

But
$$\frac{d\mathrm{J}}{dy}=\frac{\pi}{4}\ \text{when } y=0.$$

$$\therefore \qquad \mathrm{B}-\mathrm{D}=\frac{\pi}{4};\ \text{or } \mathrm{B}=-\frac{\pi}{4}.$$

$$\therefore \qquad \frac{d\mathrm{J}}{dy}=\frac{\pi}{4}ye^{-y}+\frac{\pi}{4}e^{-y}$$

$$\therefore \qquad \int_0^\infty \frac{\cos x}{(1+x^2)^2}\,dx=\frac{\pi}{4}e^{-1}+\frac{\pi}{4}e^{-1}=\frac{\pi}{2}e^{-1}$$

$$\therefore \qquad \int_0^\infty \frac{\cos 2x}{(1+x^2)^2}\,dx=\frac{\pi}{2}e^{-2}+\frac{\pi}{4}e^{-2}=\frac{3\pi}{8}e^{-2}$$

Question 475.

(K APPUKUTTAN ERADY, M.A.):—The space bounded by the coordinate planes and the surface $(x/a)^n+(y/b)^n+(z/c)^n=1$ is filled with an elastic fluid without weight. Prove that the pressures on the curved surface reduce to a single resultant whose line of action is

$$a(x-\lambda a)=b(y-\lambda b)=c(z-\lambda c),$$

where $\lambda=2\,\{\Gamma(2/n)\}^2\div 3\,\{\Gamma(1/n).\Gamma(3/n)\}$.

Solution by V. K. Aravamudan, B.A., R. Jagannathan and T. S. Krishna Rao.

As usual in such cases, project the surface on the three co-ordinate planes and find the resultant pressures along three mutually perpendicular directions.

Since the fluid is elastic and without weight

$$\frac{dp}{p}=0,\ \text{or } p=\text{const.}=k(\text{say}),$$

First, projecting on the xy plane, the resolved pressure Z along ZO is $k\iint dx\,dy$, the integral being taken for all positive values of x and y given by the limiting equation $(x/a)^n+(y/b)^n=1$.

Hence its value is

$$k\,a\,b\frac{\left(\Gamma\frac{1}{n}\right)^2}{\Gamma\left(1+\frac{2}{n}\right)}$$

The co ordinates of the point through which it acts are given by

$$\frac{\iint x\,dx\,dy}{\iint dx\,dy};\ \frac{\iint y\,dx\,dy}{\iint dx\,dy};\ 0.$$

Using Dirichlet's integrals, these are seen to be

$$\frac{a^2b\,\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{1}{n}\right)\Gamma\left(1+\frac{2}{n}\right)}{ab\,\Gamma\left(\frac{1}{n}\right)^2\Gamma\left(1+\frac{3}{n}\right)},\quad \frac{a\,b^2\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{1}{n}\right)\Gamma\left(1+\frac{2}{n}\right)}{ab\,\Gamma\left(\frac{1}{n}\right)^2\Gamma\left(1+\frac{3}{n}\right)},\ 0;$$

i.e.,
$$\frac{2}{3}\frac{\left(\Gamma\frac{2}{n}\right)^2}{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{3}{n}\right)}a,\ \frac{2}{3}\frac{\left(\Gamma\frac{2}{n}\right)}{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{3}{n}\right)}b,\ 0;$$

i.e.,
$$a\lambda,\ b\lambda,\ 0.$$

Similarly, the other components are

$$X=\mu/a, \text{ acting at } (o,\ b\lambda,\ c\lambda),$$
$$Y=\mu/b, \text{ acting at } (a\lambda,\ o,\ c\lambda),$$

where
$$\mu=\frac{kabc\ \Gamma\left(\frac{1}{n}\right)^2}{\Gamma\left(1+\frac{2}{n}\right)}$$

Now, since each of these passes through $(a\lambda, b\lambda, c\lambda)$, the resultant is a single pressure whose direction cosines are proportional to x, y, z, *i.e.*, to $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$.

Thus its line of action is $a(x-a\lambda)=b(y-b\lambda)=c(z-c\lambda)$.

Question 482.

(A. NARASINGA RAO) :—If d_r be the number of divisors of integer r (unity excluded), shew that

$$L(d_1+d_2+\ldots d_n) \to n \log n, \text{ as } n \to \infty .$$

Solution by A. C. L. Wilkinson, M.A., F.R.A.S.

The following asymptotic formula is given by Dirichlet, *Journal de Liouville*, 1856, p. 359.

$$d_1+d_2+d_3+d_n = n \log n + (2C-1)\, n + \ldots\ldots$$

where C is Euler's constant. A proof will be found in Cesáro: *Calcolo Infinitesimale*, p. 96.

Question 483.

(V. V. S. NARAYAN):—Construct a triangle ABC being given the angle A, the side AB, and the distance of the middle point of AC from the symmedian point.

Solution by N. P. Pandya.

Let K be the symmedian point and E the midpoint of AC, and let EK $=p$ (a given length.)

By trilinear coordinates, E is the point $\left(\frac{\Delta}{a}, o, \frac{\Delta}{c}\right)$; K is the point

$$\frac{2a\Delta}{a^2+b^2+c^2}, \frac{2b\Delta}{a^2+b^2+c^2}, \frac{2c\Delta}{a^2+b^2+c^2}.$$

$$\therefore\ p^2 = \frac{abc}{4\Delta^2}\left\{a \cos A\left(\frac{\Delta}{a} - \frac{2a\Delta}{a^2+b^2+c^2}\right)^2 + b \cos B\left(\frac{-2b\Delta}{\Sigma a^2}\right)^2 \right.$$
$$\left. + c \cos C\left(\frac{\Delta}{c} - \frac{2c\Delta}{a^2+b^2+c^2}\right)^2\right\}$$

$$= \frac{b^4}{4\,(b^2+c^2-bc\cos A)^2}(3c^2\cos^2 A + b^2 + c^2 - 4bc\cos A),$$

after sufficient reduction.

Thus, since p, c and A are known, b can be found from the above, and the required triangle is therefore determined.

Question 484.

(K. APPUKUTTAN ERADY M.A.):—If $\phi(xyz) \equiv (abcfgh)(x,y,z)^2$, prove that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\phi} \cos\,(px+qy+rz)\, dx\, dy\, dz = \frac{\pi^{\frac{3}{2}}}{\Delta^{\frac{1}{2}}}\, e^{-\lambda}.$$

where Δ is the discriminant of ϕ and $\lambda = \frac{(ABCFGH)(pqr)^2}{\Delta}$

Solution by R. Jagannathan.

Consider the conic $\phi=0$, and the straight line $px+qy+rz=0$. It is easy to see that by changing the triangle of reference to a self conjugate Δ having as side $Z=0$ the line $px+qy+rz=0$, ϕ transforms into $\alpha X^2+\beta Y^2+\gamma Z^2$ and $px+qy+rz$ into $2bZ$, say.

The integral in the question is therefore transformed into

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx \int_{-\infty}^{\infty} e^{-\beta y^2} dy \int_{-\infty}^{\infty} e^{-\gamma z^2} \cos.\ 2bz\ dz.$$

Now $$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \frac{1}{\sqrt{\alpha}}\int_{-\infty}^{\infty} e^{-z^2} dz, (\text{putting } \sqrt{\alpha}\cdot x=z) = \frac{\sqrt{\pi}}{\sqrt{\alpha}}.$$

$$\int_{-\infty}^{\infty} e^{-\beta y^2} dy = \frac{\sqrt{\pi}}{\sqrt{\gamma}}, \text{ similarly.}$$

Also, $$\int_{-\infty}^{\infty} e^{-\gamma z^2} \cos 2bz dz = \frac{\sqrt{\pi}}{\sqrt{\gamma}}\cdot e^{-\frac{b^2}{\gamma}}$$ (WILLIAMSON : *Integ. Calc. p*, 152.)

We have, further, the invariant relations

$\Delta=\alpha\beta\gamma$ and $Ap^2+Bq^2+Cr^2+2Fqr+2Gqr+2Hpq=4b^2\alpha\beta$.

$$\therefore \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(\alpha x^2+\beta y^2+\gamma z^2)} \cos 2bz\ dxdydz$$

$$=\frac{\pi^{3/2}}{\sqrt{\alpha\beta\gamma}}.e^{\frac{-6^2}{\gamma}} = \frac{\pi^{3/2}}{\sqrt{\alpha\beta\gamma}}.e^{\frac{-46^2\alpha\beta}{4\alpha\beta\gamma}}$$

$$=\frac{\pi^{3/2}}{\Delta^{\frac{1}{2}}}.e^{-\lambda}, \text{ where } \lambda=\frac{(ABCFGH)(pqr)^2}{\Delta}.$$

From the invariant character of Δ and λ, it is at once clear that by retransforming from X ,Y, Z to (x, y, z) our original coordinates, we get the general theorem asked for

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\phi}\cos (px+qy+rz)\ dx\ dy\ dz$$

$$=\frac{\pi^{3/2}e^{-\lambda}}{\Delta^{\frac{1}{2}}}$$

Question 486.

(A. N. RAGHAVACHAR, M. A.) :—If α, β, γ, δ be the roots of $x^4+px^3+qx^2+rx+s=0$, form the equation whose roots are

$$\frac{\beta\gamma(\alpha+\delta)-\alpha\delta(\beta+\gamma)}{\beta+\gamma-\alpha-\delta}, \text{ etc, etc.}$$

Solution by the Proposer.

Let $\rho=\dfrac{\beta\gamma(\alpha+\delta)-\alpha\delta(\beta+\gamma)}{\beta+\gamma-\alpha-\delta}$; then, by cross multiplication and tranposition, we have

$$\therefore \qquad (\alpha+\delta)(\beta\gamma+\rho)=(\beta+\gamma)(\alpha\delta+\rho).$$

$$\therefore \qquad \frac{\alpha+\delta}{\rho+\alpha\delta}=\frac{\beta+\gamma}{\rho+\beta\gamma}=-\frac{p}{2\rho+\alpha\delta+\beta\gamma};$$

$$=\frac{(\alpha+\delta)\beta\gamma+(\beta+\gamma)\alpha\delta}{\rho(\beta\gamma+\alpha\delta)+2\alpha\beta\gamma\delta}=-\frac{r}{\rho(\beta\gamma+\alpha\delta)+2s}.$$

Thus $\qquad (\beta\gamma+\alpha\delta)(p\rho-r)=2(\rho r-ps) \qquad \ldots \qquad \ldots \qquad (1)$

Again $\qquad \rho^2=\dfrac{(\Sigma\alpha\beta\gamma)^2-4\alpha\beta\gamma\delta(\alpha+\delta)(\beta+\gamma)}{(\Sigma\alpha)^2-4(\alpha+\delta)(\beta+\gamma)}$

$$=\frac{r^2-4\alpha\beta\gamma\delta(\alpha+\delta)(\beta+\gamma)}{p^2-4(\alpha+\delta)(\beta+\gamma)}$$

$$=\frac{(r^2-4sq)(p\rho-r)+8s(\rho r-ps)}{(p^2-4q)(p\rho-\gamma)+8(\rho r-ps)}, \text{ from (1)}$$

In other words, ρ is a root of the equation

$$\rho^3(p^3-4pq+8r)-\rho^2(rp^2-4qr+8ps)+\rho(4pqs-pr^2-8rs)$$
$$+(r^3-4qrs+8ps^2)=0.$$

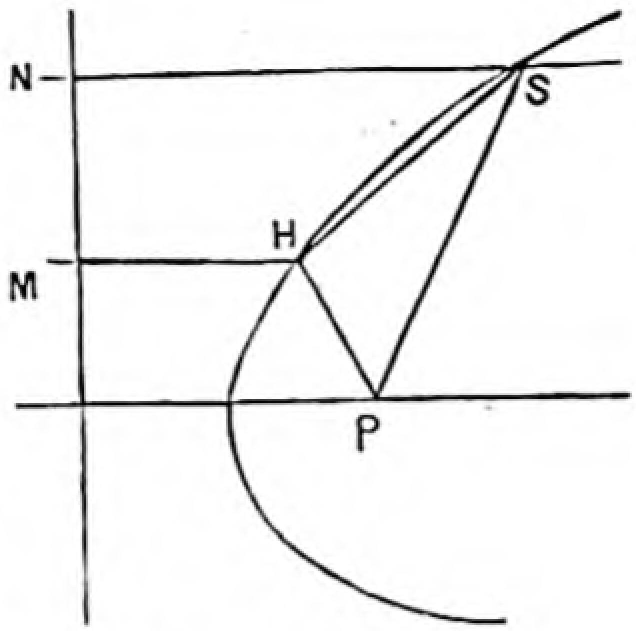
Question 487.

(N. P. PANDYA) :—If P be a point on a hyperbola whose foci are S and H, show that its asymptotes are parallel to the axes of the parabola passing through S, H and having P for their common focus.

Solutions (1) *by T. P. Trivedi, M.A., L.L.B.,* (2) *by R. Srinivasan, M.A., D. Krishna Murti, V. K. Aravamudan, B.A., and A. Norasinga Rao*; (3) *by N. Sankara Aiyar, M.A.*

(1) Drawing any one parabola through S and H having P for the focus, we see that the inclination of the axis of the parabola to SH being θ,

$$\cos\theta=\frac{SN-HM}{SH}=\frac{SP-HP}{SH}=\frac{2a}{2ae},$$



where e is the eccentricity of the hyperbola.

$$\therefore \qquad \cos\theta=\frac{1}{e}=\frac{a}{\sqrt{a^2+b^2}}. \quad \therefore \tan\theta=\frac{b}{a}.$$

But the asymptotes of the hyperbola are inclined to SH on eithers side at an angle $\tan^{-1}(b/a)$; hence these asymptotes are parallel to the axe of the two parabolas which can be drawn with the given data.

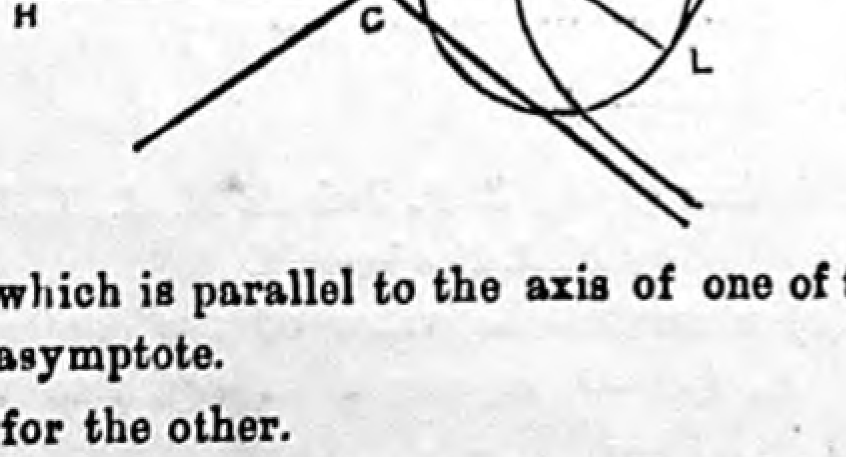
(2) Let PG be the normal at P. The directrix of the parabolas are the common tangents of circles described with centres S and H, and radii SP and HP.

Clearly G is the external centre of similitude of these circles.

Let GL be the tangent to the circle (S) at L.

We know that SG : SP $=e=$ SG : SL.

$$\therefore \qquad \angle LSG=\cos^{-1}\left(\frac{1}{e}\right)$$



Thus SL (which is parallel to the axis of one of the parabolas) is parallel to one asymptote.

Similarly, for the other.

(3) Let P be $a \sec \alpha$, $b \tan \alpha$. It. Then the parabolas will be

$$(a \sec \alpha-x)^2+(b \tan \alpha-y)^2=(x \cos \beta+y \sin \beta-p)^2.$$

Since these pass through S and H we get

$$a^2(\sec\alpha-e)^2+b^2\tan^2\alpha=(ae\cos\beta-p)^2$$
$$a^2(\sec\alpha+e)^2+b^2\tan^2\alpha=(ae\cos\beta+p)^2.$$

Subtracting

$$4a^2e\sec\alpha=4aep\cos\beta.$$

i.e., $a\sec\alpha=p\cos\beta$, or $p=a\sec\alpha\sec\beta$. Substituting this value of p, we get

$$a^2(\sec\alpha-e)^2+b^2\tan^2\alpha=(ae\cos\beta-a\sec\alpha\sec\beta)^2,$$

i.e., $\sec^2\alpha-2e\sec\alpha+e^2+e^2\tan^2\alpha-\tan^2\alpha$
$$=e^2\cos^2\beta-2e\sec\alpha+\sec^2\alpha\sec^2\beta$$

i.e., $1+e^2\sec^2\alpha=e^2\cos^2\beta+\sec^2\alpha\sec^2\beta^4$

i.e., $\sec^4\beta\sec^2\alpha-\sec^2\beta-e^2\sec^2\beta\sec^2\alpha+e^2=0.$

i.e., $(\sec^2\beta-e^2)(\sec^2\alpha\sec^2\beta-1)=0.$

i.e., $\sec^2\beta=e^2$, the other solution being inadmissible.

i.e., $\sec\beta=\pm e.$

The axis which is perpendicular to the directrix $x\cos\beta+y\sin\beta-p=0$, therefore, makes an angle $\sec^{-1}(\pm e)$ with the axes of reference and is therefore parallel to an asymptote of the given hyperbola.

Question 488.

(Zero):—Prove that the roots of the determinant equation of the n^{th} degree

$$\begin{vmatrix} x, & 1, & 0, & 0. & 0 & \dots & \dots \\ 1, & x, & 1, & 0, & 0 & \dots & \dots \\ 0, & 1, & x, & 1, & 0 & \dots & \dots \\ 0, & 0, & 1, & x, & 1 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}=0,$$

are $2\cos\frac{k\pi}{n+1}(k=1, 2, 3\dots n)$.

Additional Solutions (1) *by T. P. Trivedi M. A., L. L. B., and* (2) *by K. J. Sanjana M. A.*

(1) Let $f_n(x)$ represent the value of the determinant of the n^{th} order, and let $F_n(x)=0$ represent an equation whose roots are $2\cos\frac{k\pi}{n+1}(k=1, 2\dots n)$. We shall show that the expressions $F_n(x)$ and $f_n(x)$ are identical.

Expanding the determinant in terms of the elements of the first row, we get

$$f_n(x) = x\begin{vmatrix} x, & 1, & 0, & 0 & \dots & \dots \\ 1, & x, & 1, & 0 & \dots & \dots \\ 0, & 1, & x, & 1 & \dots & \dots \\ 0, & 0, & 1, & x, & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix} - 1\begin{vmatrix} 1, & 1, & 0, & 0 & \dots & \dots \\ 0, & x, & 1, & 0 & \dots & \dots, \\ 0, & 1, & x, & 1 & \dots & \dots \\ 0, & 0, & 1, & x, & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

$$= xf_{n-1}(x) - f_{n-2}(x).$$

i.e., $$f_n(x) + f_{n-2}(x) = xf_{n-1}(x).$$

Again, if y represents $\cos\dfrac{k\pi}{n+1} + i\sin\dfrac{k\pi}{n+1}$, then $y^{2n+2} - 1 = 0$.

Dividing this by $y^2 - 1$ and writing the equation as

$$y^n + y^{n-2} + y^{n-4} + \dots + \dots\dots\dots\dots y^{-n+2} + y^{-n} = 0,$$

we see that $F_n(x)$ is obtained by putting $y + \dfrac{1}{y} = x$ in this, since $y + \dfrac{1}{y} = 2\cos\dfrac{k\pi}{n+1}$.

Again, we have identically.

$$(y^n + y^{n-2} + y^{n-4} + \dots\dots y^{-n+2} + y^{-n}) + (y^{n-2} + y^{n-4} + \dots + y^{-n+4} + y^{-n+2})$$
$$= (y + y^{-1})(y_u^{-1} + y^{n-3} + \dots y^{-1+3} + y^{-n+1})$$

On substituting $y + y^{-1} = x$, we have the same relation as before; *viz.*,

$$F_n(x) + F_{n-2}(x) = x\,F_{n-1}(x)$$

Now, taking the determinants of the first three orders, we get the expressions

$$x, \quad \begin{vmatrix} x, & 1 \\ 1, & x \end{vmatrix} = x^2 - 1, \quad \begin{vmatrix} x, & 1, & 0 \\ 1, & x, & 1 \\ 0, & 1, & x \end{vmatrix} = x^3 - 2x;$$

and we know that the roots of $x=0$, $x^2-1=0$ and $x^3-2x=0$ are $2\cos\dfrac{k\pi}{2}(k=1)$; $2\cos\dfrac{k\pi}{3}(k=1 \text{ or } 2)$ and $2\cos\dfrac{k\pi}{4}(k=1, 2 \text{ or } 3)$.

Hence we have the same solutions for $f_1(x)$ and $F_1(x)$; $f_2(x)$ and $F_2(x)$; $f_3(x)$ and $F_3(x)$; hence by using the above relations, we see that ultimately $f_n(x)$ and $F_n(x)$ are identical.

(2) Putting the determinant $= f_n(x)$, we can readily show that $f_n(x) = x\,f_{n-1}(x) - f_{n-2}(x)$, or

$$f_n(x) + f_{n-2}(x) = x\,f_{n-1}(x).$$

Now it is well known that $\frac{\sin n\theta}{\sin\theta}$ is a function of $2\cos\theta$ of degree $n-1$; putting $\xi=2\cos\theta$ and denoting the function by F, we have

$$F_n(\xi)+F_{n-2}(\xi)=\frac{\sin(n+1)\theta+\sin(n-1)\theta}{\sin\theta}=\frac{2\sin n\theta\cos\theta}{\sin\theta}$$
$$=\xi\, F_{n-1}(\xi);$$

so that three consecutive F's are connected in exactly the same way as three consecutive f's.

Also, $F_1(\xi)=0$ gives $\frac{\sin 2\theta}{\sin\theta}=0$, or $2\cos\theta=0$, *i.e.*, $\theta=\frac{1.\pi}{2}$;

$F_2(\xi)=0$, gives $\frac{\sin 3\theta}{\sin\theta}=0$, or $4\cos^2\theta-1=0$, *i.e.*, $\theta=\frac{(1,2)\pi}{3}$;

and $F_3(\xi)=0$ gives $\frac{\sin 4\theta}{\sin\theta}=0$ or $\cos 2\theta\cos\theta=0$, *i.e.*, $\theta=\frac{(1,2,3)\pi}{4}$.

The first three solutions of $f_n(x)=0$, give $x=0$, $x=\pm1$, and $x\pm\sqrt{2}$, which are the doubled cosines of the angles found above. Hence, the first three solutions coinciding and formula of recurrence being the same, it is evident that all succeeding solutions of the equations $f_n(x)=0$, and $F_n(\xi)=0$, will coincide.

The solution of $F_n(\xi)=0$ is that of $\frac{\sin(n+1)\theta}{\sin\theta}=0$.

If $\sin(n+1)\theta=0$, we have $\theta=\frac{(0,1,2,\ldots,n)\pi}{n+1}$, and omitting θ which correspond to $\sin\theta$ in the denominator, we see that the n values of ξ, and therefore of x, are

$$2\cos\frac{k\pi}{n+1},\ [k=1,2,\ldots,n].$$

I find from some notes that this problem was proposed by Mr. Tits in *Mathesis*, but have no recollection when and on what lines the solution was given.

Question 490.

(R. Srinivasan, M.A.):—Shew that

$$\sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{(2n+1)\Gamma(n+\frac{3}{2})}=\sum_{n=0}^{\infty}\left\{\frac{4}{\sqrt{\pi}}\frac{(-1)^n}{(2n+1)^2}\right\}$$

Solution (1) *by S. Ramanujan*, (2) *by S. Krishnaswami Aiyangar.*

(1) The question should be as printed above.

$$\text{The left side}=\frac{2}{\sqrt{\pi}}\left\{\frac{1}{1^2}+\frac{2}{3^2}+\frac{2\cdot4}{3\cdot5^2}+\frac{2\cdot4\cdot6}{3\cdot5\cdot7^2}+\cdots\right\}$$

$$=\frac{2}{\sqrt{\pi}}\int_0^1\frac{\sin^{-1}x}{x\sqrt{1-x^2}}dx,$$

$$=\frac{2}{\sqrt{\pi}}\int_0^{\frac{\pi}{4}}\frac{4+\theta.\cos 2\theta}{\sin 2\theta\cos 2\theta}d\theta, \text{ (putting } x=\sin 2\theta)$$

$$=\frac{4}{\sqrt{\pi}}\int_0^1\frac{1}{y}\tan^{-1}y\,dy \text{ (put } y=\tan\theta)$$

$$=\frac{4}{\sqrt{\pi}}\left\{\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\dots\right\}=\sum_0^\infty\left\{\frac{4}{\sqrt{\pi}}\frac{(-1)^n}{(2n+1)^2}\right\}.$$

(2) $$\sum_0^\infty\frac{\Gamma(n+1)}{(2n+1)\,\Gamma(n+\frac{3}{2})}=\frac{2}{\sqrt{\pi}}\sum_0^\infty\int_0^{\frac{\pi}{2}}\frac{\sin^{2n+1}\theta}{2n+1}d\theta$$

$$=\frac{2}{\sqrt{\pi}}\int_0^{\frac{\pi}{2}}\left\{\sin\theta+\frac{\sin^3\theta}{3}+\frac{\sin^5\theta}{5}+\dots\dots\right\}d\theta$$

$$=\frac{2}{\sqrt{\pi}}\int_0^{\frac{\pi}{2}}\left\{\tanh^{-1}(\sin\theta)\right\}d\theta.$$

$$=\frac{2}{\sqrt{\pi}}\int_0^\infty x\operatorname{sech}x\,dx, \text{ changing } \sin\theta \text{ to } \tanh x.$$

$$=\frac{2}{\sqrt{\pi}}\int_0^\infty 2xe^{-x}(1+e^{-2x})^{-1}dx.$$

$$=\frac{4}{\sqrt{\pi}}\int_0^\infty\left\{xe^{-x}-xe^{-3x}+xe^{-5x}\dots\dots\right\}d$$

$$=\frac{4}{\sqrt{\pi}}\left\{\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\dots\dots\dots+(-)^n\frac{1}{(2n+1)^2}+\dots\dots\right\}$$

$$=\sum_0^\infty\left\{\frac{4}{\sqrt{\pi}}(-)^n\frac{1}{(2n+1)^2}\right\}$$

Question 491.

(M. T. NARANIENGAR,) :—In any triangle prove that a circumconic passing through the ends of a diameter of the maximum inscribed ellipse touches the ellipse.

Solution by A. Narasinga Rao.

Project the triangle into an equilateral triangle. Then the maximum inscribed ellipse projects into the incircle of the equilateral triangle. Also the isogonal transformation of the incircle is the three-

cusped hypocycloid with the cusps at the vertices, and any line through the centre of the triangle transforms into a rectangular hyperbola through the vertices.

Hence the theorem reduces to :

If a rectangular hyperbola through the vertices of the triangle cuts the tricusp at P and Q, then PQ touches the tricusp.

This property has been proved in the Journal. (*Vide*: Vol, V. p. 86.)

Question 492.

(K. V. Anantnarayan Sastri, B.A. :—Expand $\theta \tan\frac{\theta}{2}$ in powers of $\sin\theta$.

Solution (1) *by T. P. Trivedi, M.A., LL.B., and S. Ramanujan* (2) *by R. Vythynathaswamy.*

$$(1)\quad \theta\tan\frac{\theta}{2}=\frac{\theta\sin\theta}{1+\cos\theta}=\frac{x\sin^{-1}x}{1+\sqrt{1-x^2}},$$

$$=\frac{\sin^{-1}x(1-\sqrt{1-x^2})}{x}, \text{ if } x=\sin\theta.$$

Now $$\frac{\sin^{-1}x}{x}=1+\tfrac{1}{2}\cdot\frac{x^2}{3}+\frac{1\cdot3}{2\cdot4}\,\frac{x^4}{5}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\,\frac{x^6}{7}+\ldots$$

Again $$\frac{\sin^{-1}x}{\sqrt{1-x^2}}=x+\frac{2}{3}x^3+\frac{2\cdot4}{3\cdot5}x^5+\frac{2\cdot4\cdot6}{3\cdot5\cdot7}x^7+\ldots$$

$$\therefore\quad \frac{\sin^{-1}x\sqrt{1-x^2}}{x}=\frac{1}{x}\left(x+\frac{2}{3}x^3+\frac{2\cdot4}{2\cdot5}x^5+\frac{2\cdot4\cdot6}{3\cdot5\cdot7}x^7+\ldots,\right)(1-x^2)$$

$$=1-\frac{x^2}{3}-\frac{2}{3}\cdot\frac{x^4}{5}-\frac{2\cdot4x^6}{3\cdot5\,7}-\ldots$$

Hence

$$\frac{\sin^{-1}x(1-\sqrt{1-x^2})}{x}=\frac{x^2}{3}(1+\tfrac{1}{2})+\frac{x^4}{5}\left(\frac{2}{3}+\frac{1\cdot3}{2\cdot4}\right)$$
$$+\frac{x^6}{7}\left(\frac{2\cdot4}{3\cdot5}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}+\ldots\ldots\right)$$

Replacing x by $\sin\theta$, the result is obtained.

(2) Let $t=\tan\frac{\theta}{2}$ and $s=\sin\theta$. Then

$$t^2s-2t+s=0.$$

$\therefore$ [Edwards *Diff. Cal.* Page 457, Ex. 8 (1).]

$$\therefore\quad t=\frac{s}{2}+\frac{s^3}{3^3}+\frac{4s^5}{2\cdot2^5}+\frac{5.6}{2\cdot3}\,\frac{s^7}{2^7}+\ldots\ldots$$

and $$\theta=s+\frac{1}{2\cdot3}\frac{s^3}{}+\frac{1\cdot3}{2\cdot4}\frac{s^5}{5}+\ldots\ldots$$

By multiplying the two series, we get the expansion as given.

QUESTIONS FOR SOLUTION.

508. (S. P. Singaravelu Modeliar) :—If s_n stand for the sum of the reciprocals of the first n natural numbers, find the sum of the infinite series $s_1 - \frac{1}{2}(\frac{1}{2})^2 s_2 + \frac{1}{3}\left(\frac{1\cdot3}{2\cdot4}\right)^2 s_3 - + \ldots\ldots$

509. (K. Appu Kuttan Errady, M.A.) :—Forces act at the point (f, g, h) within the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, and are represented by the normals from the point to the surface. Show that the resultant acts along the lines whose direction cosines are proportional to

$$f\left(\frac{b^2}{a^2-b^2} + \frac{c^2}{a^2-c^2} - 1\right), \text{ etc., etc.,}$$

and find its magnitude.

510. (S. P. Singaravelu Mudaliar) :—Shew that the locus of the symmedian point of the triangle of maximum area inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{4}\cdot\left(\frac{a^2-b^2}{a^2+b^2}\right).$$

511. (S. Krishnaswami Aiyangar) :—If S and H are the foc of the maximum inscribed ellipse of the triangle of reference prove that

$$\text{AS}\cdot\text{BS}\cdot\text{CS}\cdot\text{AH}\cdot\text{BH}\cdot\text{CH} = \frac{a^2b^2c^2}{27}.$$

512. (A Narasinga Rao) :—If $n\phi(n) + (n+2)\,\phi(n+2) = \phi(n+1)$, prove that

(i) $\phi(1)x - \phi(3)x^3 + \phi(5)x^5 - \ldots\ldots = \phi(1)\sin(\tanh^{-1}x)$,

(ii) $\phi(0) - \phi(2)x^2 + \phi(4)x^4 - \ldots\ldots = \phi(0) + \phi(1)(\cos\tanh^{-1}x + 1)$.

Discuss the convergency of the series $\Sigma\phi(n)\,x^n$.

513. (Professors T. P. Trivedi and K. J. Sanjana).—Find the sum of the series

$$1 - \frac{h}{h-k+2} + \frac{h(h-1)}{(h-k+2)(h-k+3)} - \frac{h(h-1)(h-2)}{(h-k+2)(h-k+3)(h-k+4)} + \ldots,$$

where h is a positive integer and k any rational number.

514. (A. C. L. Wilkinson M.A., F.R., A.S.) :—All conics cutting a rectangular hyperbola orthogonally at all four points of intersection consist of (1) hyperbolas having the axes of the given hyperbola as

asymptotes, (2) all conics confocal with the given hyperbola, (3) two sets of ellipses each passing through two fixed points and whose centres line on one or other of the axes of the hyperbola and whose axes are parallel to the asymptotes of the hyperbola.

515. (A. C. L. WILKINSON M.A., F.R A.S.) :—ABCD is a quadrilateral. DQ, BP are any two parallel straight lines meeting AB, CD respectively in Q and P; QL parallel to CD meets BC in L and PN parallel to AB meets AD in N; prove that the middle points of AC, DL, BN are collinear.

516. (K. J. SANJANA, M. A.) :—TP, TQ tangents to a conic of centre C and focus F, cut the auxiliary circle in Y, Z, and FW is perpendicular to the chord of contact; a tangent of the conic perpendicular to TC cuts FY, FZ, FW in y, z, w, respectively. Prove that $yw=wz$, and enunciate the corresponding theorem for the parabola.

517. (T. P. TRIVEDI M.A., L.L B.)—Is there any known method of integrating completely

$$\left(\frac{d^2y}{dx^2}\right)^2\frac{d^5y}{dx^5}-\frac{d^2y}{dx^2}\frac{d^3y}{dx^3}\frac{d^4y}{dx^4}+\frac{40}{9}\left(\frac{d^3y}{dx^3}\right)^3=0,$$

which is the well-known differential equation of all conic sections?

518. (R. SRINIVASAN) :—If the centroid of the triangle formed by the normals to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the points whose eccentric angles are α, β, γ, lies on the major axis, shew that

$$3+4\Sigma\cos\alpha\Sigma\cos\beta\cos\gamma+2\Sigma\cos2\alpha+\Sigma\cos2(\alpha+\beta)=0.$$

519. (V. ANANTARAMAN) :—Given the base and the vertical angle of a triangle find the locus of the centre of the circle passing through the three excentres.

520. (R. VYTHYNATHASWAMY) :—ABCD is a quadrilateral inscribed in a circle, centre O. Denoting the simson line of A *w. r. t.* BCD by *a* and so on, shew that, if *a*, *b*, *c*, *d* intersect OA, OB, OC, OD in P, Q, R, S and themselves cointersect at T, then P, Q, R, S, T, O are cyclic.

521. (K. V. ANANTANARAYANA SASTRY, B.A.) :—If V denote the entire volume of the figure bounded by $(x/a)^{\frac{16}{17}}+(y/b)^{\frac{16}{17}}+(z/c)^{\frac{16}{17}}=1$, and A the whole area of its trace on the xy-plane, prove that

$$V=\frac{17A.c}{19\cdot35}.B\left(\frac{1}{8},\frac{1}{16}\right).$$

522. (V. V. S. NARAYAN) :—Given a parallelogram (drawn on paper) and an ungraduated straight edge, show how to trisect a given finite straight line drawn on the same paper.

523. (R. N. APTE, M.A., F.R.A.S.,) :—Find the value of

$$\iint x\,y\,z\left\{1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2\right\}dx\,dy,$$

where $z=c\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{\frac{1}{2}}$, and the integration is over the positive quadrant of $x^2/a^2+y^2/b^2=1$.

524. (S. RAMANUJAN) :—Shew that

(i) $\sqrt[3]{\cos\frac{2\pi}{7}}+\sqrt[3]{\cos\frac{4\pi}{7}}+\sqrt[3]{\cos\frac{8\pi}{7}}=\sqrt[3]{\frac{5-3\sqrt[3]{7}}{2}}$

(ii) $\sqrt[3]{\cos\frac{2\pi}{9}}+\sqrt[3]{\cos\frac{4\pi}{9}}+\sqrt[3]{\cos\frac{8\pi}{9}}=\sqrt[3]{\frac{3\sqrt[3]{9}-6}{2}}$.

525. (S. RAMANUJAN) :—Shew how to find the square roots of surds of the form $\sqrt[3]{A}+\sqrt[3]{B}$, and hence prove that

$$\sqrt{\sqrt[3]{5}-\sqrt[3]{4}}=\frac{\sqrt[3]{2}+\sqrt[3]{20}-\sqrt[3]{25}}{3};$$

and

$$\sqrt{\sqrt[3]{28}-\sqrt[3]{27}}=\frac{\sqrt[3]{98}-\sqrt[3]{28}-1}{3}.$$

526. (S. RAMANUJAN) :—If n is any positive quantity shew that $\frac{1}{n}>\frac{1}{n+1}+\frac{1}{(n+2)^2}+\frac{3}{(n+3)^3}+\frac{4^2}{(n+4)^4}+\frac{5^3}{(n+5)^5}+\dots$ Find the difference approximately when n is great.

Hence shew that

$$\frac{1}{1001}+\frac{1}{1002^2}+\frac{3}{1003^3}+\frac{4^2}{1004^4}+\frac{5^3}{1005^5}+\dots<\frac{1}{1000} \text{ by } 10^{-440} \text{ nearly.}$$

527. (A. NARASINGA RAO) :—A family of quartics touches each side of triangle ABC at A, B, C and passes through four given points. Prove that only two members of the family will touch the circumcircle of ABC.

528. (A. N. RAGHAVACHAR M.A.) :—If (n, r) denote the sum of the products r together of the n natural numbers, find the value of

$$\frac{(2n-1,n)}{\lfloor 2n}-\frac{(2n-2,n)}{\lfloor 1\lfloor 2n-1}+\frac{(2n-3,n)}{\lfloor 2\lfloor 2n-2}+\cdots\frac{(-)^{n-1}(n,n)}{\lfloor n-1\lfloor n+1}.$$

529. (N. P. PANDYA) :—In a triangle ABC, D, E, F are the mid-points of BC, CA, AB. If AX, BY, CZ are perpendiculars on the sides of DEF, shew geometrically that DX, EY, FZ are concurrent at the symmedian point of ABC.

530. (N. SANKARA AIYAR M.A.) :—Solve in integers

$$\frac{x^2+17}{y+9}=\text{a perfect square}.$$

531. (N. SANKARA AIYAR, M.A.) :—Integrate

$$\int_0^\infty \frac{\sin xy}{x(1+x^2)^3}dx.$$

A List of Periodicals Received.

(From 16th November 1913 to 15th January 1914.)

1. Acta Mathematica, Vol. 37, No. 1.
2. Annals of Mathematics, Vols. 15, No. 2 December 1913.
3. Astrophysical Journal, Vol. 38, Nos. 3 & 4, October and November 1913.
4. Bulletin of the American Mathematical Society, Vol. 20, Nos. 2 & 3, November and December 1913.
5. Bulletin des Sciences Mathematiques, Vol. 37, November and December 1913.
6. Crelle's Journal, Vol. 143, No. 4, October 1913.
7. Educational Times, December 1913. and January 1914 (6 copies.)
8. L'Education, Mathematique, Vol. 16, Nos. 1. 2, 3, 4, 5 & 6.
9. Fortschritte des Mathematik, Vol. 42, No. 1.
10. L'Intermediaire des Mathematiciens, Vol. 20, Nos. 9 & 10, September and October 1913.
11. Journal de Mathematiques, Elementaires, Vol. 38, Nos. 2, 3, 4, 5 & 6.
12. Mathematical Gazette, Vol. 7 Nos. 107 & 108, October and December 1913, (3 copies.)
13. Mathematische Annalen, Vol. 74, Nos. 3 and 4, October and December 1913.
14. Messenger of Mathematics, Vol. 43, No. 6, October 1913.
15. Monthly notices of the Royal Astronomical Society, Vol. 73, No. 9, Supplementary.
16. Philosophical Magazine, Vol. 26, Nos. 155 and 156, November and December 1913.
17. Popular Astronomy, Vol. 21, Nos. 9 and 10, November and December, 1913, (3 copies.)
18. Proceedings of the London Mathematical Society, Vol. 12, No. 7, November 1913.
19. Proceedings of the Royal Society of London, Vol. 89, No. 611, November, 1913.
20. Revue de Mathematiques Speciales, Vol. 24, Nos. 2 and 3, November, and December, 1913.
21. School Science and Mathematics, Vol. 13, Nos. 8 and 9, November and December 1913, (3 copies.)
22. Transactions of the Cambridge Philosophical Society, Vol. 22, No. 3 November 1913.
23. Transactions of the Royal Society of London, Vol. 213, No. 504.

The Indian Mathematical Society.

(Founded in 1907 for the Advancement of Mathematical Study, and Research in India.)

THE JOURNAL

OF THE

Indian Mathematical Society

Vol. VI.] APRIL 1914. [No. 2. No. 2

EDITED BY

M. T. NARANIENGAR, M. A.,

Hony. Joint Secretary,

WITH THE CO-OPERATION OF

Prof. R. P. PARANJPYE, M.A., B.Sc. Prof. A. C. L. WILKINSON, M.A., F.R.A.S.

and others.

Madras

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CONTENTS.

The Journal is open to contributions from members as well as subscribers. The editors may also accept contributions from others.

Contributors will be supplied, if so desired, with extra copies of their contributions at net cost.

All contributions should be written legibly on one side only of the paper, and all diagrams should be given in separate slips.

All communications intended for the Journal should be addressed to the Hony. Joint Secretary, M. T. NARANIENGAR, M.A., Mallesvaram, Bangalore.

All business communications regarding the Journal should be addressed to the Hony. Asst. Secretary, P. V. SESHU AIYAR, B.A., L.T., 37, Venkatachala Chetty Street, Triplicane, Madras.

Enquiries by intending members and all other communications may be addressed to the Hony. Joint Secretary, D. D. KAPADIA, M.A., B.Sc., 322, Main Street, Poona.

THE JOURNAL

OF THE

Indian Mathematical Society.

Vol. VI.] APRIL 1914. [No. 2.

PROGRESS REPORT.

The following gentlemen have been elected as members of the Society :—

(1) Mr. L. N. Subrahmaniam, M.A., L.T, Acting Professor of Mathematics, S. P. G. College, Trichinopoly;

(2) Mr. R. Mahadevan, B.A., Extra Assistant Supt., Bharati Vilas, Basvangudi, Bangalore; and

(3) Mr. R. Littlehailes, M.A., Professor of Mathematics, Presidency College, Madras.

2. The following books have been received from the publishers :—

(1) *Elementary Statics*—by Dr. R. S. Heath, Clarendon Press, Oxford, 1913, 4/6;

(2) *A General Course of Pure Mathematics*—by A. L. Bowley, Clarendon Press, Oxford, 1913, 7/6;

(3) *Cours de Mecanique, Tome I*—by Leon LeCornu, Gauthier-Villars, Paris, 1914, 18 frs.;

(4) *Introduction Geometrique* a Quelques Theories Physiques-by Emile Borel, Gauthier-Villars, Paris, 1914, 5 frs.;

(5) *Les Recents Progres du Systeme Metrique* (being the report presented to the Fifth General Conference for weights and measures)—by Ch. Ed. Guillaume, Gauthier-Vil. Paris, 1914, 3 frs.

(6) *Annuaire pour L'an 1914*—publie par de Bureau des Longitudes, (avec des Notices Scientifiques), Gauthier, Vil., Paris, 1. 5 frs.

(7) *Projective Geometry*—by Hatton, Cambridge University Press, 1913, 10/6;

(8) *Exercises in Mathematics*—by D. B. Mair, Macmillan & Co., 1914, 4/6 ;

(9) *First Year Course in Mathematics*—by K. J. Sanjana, and J. Cooper, Bombay, 1914, Rs. 1-12-0 ;

(10) *Panjab University Calendar*—for 1913-1914.

(11) *A Binary Canon*—showing the Residues of Powers of 2 for divisors under 1COO, and indices to Residences—compiled by Lt. Col. A. Cunningham, (under the auspices of the British Association Committee), London 1900.

3. The Audited Balance sheet for 1913, and the Budget for 1914 are published herewith for the information of the General Body.

POONA,
31st March 1914.

D. D. KAPADIA,
Hony. Joint Secretary.

THE INDIAN MATHEMATICAL SOCIETY.

Balance Sheet for 1913.

Receipts.	Rs.	A.	P.	*Expenditure.*	Rs.	A.	P.
To Balance from 1912.	240	3	7	By Books and Journals	319	4	0
„ Subscriptions current	1247	8	0	„ Library charges ...	314	0	0
„ do. arrears.	508	8	0	„ Journal Printing...	533	7	0
„ do. life ...	150	0	0	„ Ordinary working expenses... ...	234	11	10
„ do. Journal	215	7	0	„ Balance to 1914 ...	970	1	9
„ Miscellaneous ...	9	14	0				
TOTAL Rs....	2,371	8	7	TOTAL Rs....	2,371	8	7

MADRAS,
7th February 1914.

(Sd.) C. POLLARD, M.A.
Hony. Treasurer.

I have examined the Treasurer's books and vouchers and the monthly statements submitted by the Secretary, Assistant Secretary and Assistant Librarian, and declare the above accounts to be correct.

Madras, 7th February 1914.

(Sd.) K. AMRITA ROW, M.A.,
Auditor.

Budget for 1914.

Receipts.	Rs.	A.	P.	*Expenditure.*	Rs.	A.	P.
To Balance brought forward, less £12/19/9 due to Book-sellers.	776	2	6	By Books and Journals	800	0	0
„ Subscriptions current	1,300	0	0	„ Library charges...	350	0	0
„ do. arrears.	500	0	0	„ Journal Printing.	550	0	0
„ do. Journal.	150	0	0	„ Ordinary working expenses... ...	250	0	0
„ Miscellaneous ...	3	13	6	„ Investment for Capital	500	0	0
				„ Balance to 1915 ...	280	0	0
TOTAL Rs....	2,730	0	0	TOTAL Rs....	2,730	0	0

Madras, 7th February 1914.

(Sd.) C. POLLARD, M.A.,
Hon. Treasurer.

On the Bicircular Quartic.

By A. C. L. Wilkinson, M.A., F.R.A.S.

1. A fundamental theorem of the general bicircular quartic, given by Casey, is the following:—

A bicircular quartic can be generated in four and only four ways as the envelope of a circle which cuts a fixed circle orthogonally and whose centre moves on a fixed central conic. This fixed conic is called a focal conic and the four focal conics corresponding to the four methods of generating the quartic are confocal, their foci being the double foci of the quartic.

The fixed circle will also be called a *circle of inversion*, since the bicircular quartic inverts into itself with respect to any one of these four circles.

Basset's *Elementary Treatise on Cubic and Quartic Curves*, Chap. IX gives an account of these curves. I shall refer to this work by the name of the author simply.

When the fixed conic is a parabola we obtain the Circular Cubics, and when the fixed conic is a circle we obtain the Cartesians, including the so-called Oval of Descartes.

2. Following Salmon's *Higher Plane Curves*, Chapter VI, I consider first the general binodal quartic.

Take for triangle of reference a triangle having the two nodes for two of its vertices; let $z=0$ be the side joining these vertices. Assume that the other vertex of the triangle does not lie on the quartic. The general equation to the binodal is:

$$x^2y^2+2xyz(lx+my)+z^2(ax^2+by^2+cz^2+2hxy+2gxz+2fyz)=0,$$

where c is not zero.

Identify the left hand side with

$$(H_1xy+G_1xz+F_1yz)(H_2xy+G_2xz+F_2zy)$$
$$-C(z^2+Hxy+Gxz+Fyz)^2=0. \qquad \ldots (1)$$

The following equations arise for determining the ten constants:

$$C=-c,\ F_1G_2+F_2G_1-2CH-2CGF=2h$$
$$H_1H_2-CH^2=1,\ F_1H_2+F_2H_1-2CHF=2m$$
$$G_1G_2-CG^2=a,\ H_1G_2+H_2G_1-2CHG=2l$$
$$F_1F_2-CF^2=b,\ 2CG=-2g,\ 2CF=-f.$$

We have to see that these equations are consistent and sufficient or the determination of H_1, G_1, F_1 etc.

Now C, G, F are at once determined uniquely and we may re-write the other equations, as follows:

$$H_1H_2=1+CH^2,\ F_1G_2+F_2G_1=2h+2CH+2CGF,$$
$$G_1G_2=a+CG^2,\ F_1H_2+F_2H_1=2m+2CHF,$$
$$F_1F_2=b+CF^2,\ H_1G_2+H_2G_1=2l+2CHF,$$

which are precisely the conditions that

$$(1+CH^2)x^2+(a+CG^2)y^2+(b+CF^2)z^2+2(l+CHG)xy$$
$$+2(m+CHF)xz+2(h+CH+CFG)yz=0$$

may break up into the factors $H_1x+G_1y+F_1z$ and $H_2x+G_2y+F_2z$.

Thus since C, G, F are known, we have for H the equation

$$\begin{vmatrix} 1+CH^2, & l+CGH, & m+CHF \\ l+CGH, & a+CG^2, & h+CH+CGF \\ m+CHF, & h+CH+CGF, & b+CF^2 \end{vmatrix}=0.$$

This is a biquadratic for H, and corresponding to each value of H derived from this equation the factors $H_1x+G_1y+F_1z$ and $H_2x+G_2y+F_2z$ are uniquely determined, except for an arbitrary constant multiplier which does not affect the product.

From this the following theorem results:

The binodal quartic may be exprcesed in the form (1) in precisely four different ways and in general these are distinct.

More generally the quartic can be reduced to the form

$$(C_1z^2+H_1xy+G_1xr+F_1yz)\ (C_2z^2+H_2xy+G_2xz+F_2yz)$$
$$-(Cz^2+Hxy+Gxz+Fyz)^2=0,$$

in an infinite number of ways and it is quite easy to reduce all these to the fundamental form (1).

For, writing this $UV=W^2$, where $U=C_1z^2+++$, $V=C_2z^2+++$ $W=Cz^2+++$, we have

$$(W+\lambda U)^2=U(V+2\lambda W+\lambda^2U),$$
$$\{W+\lambda U+\mu(V+2\lambda W+\lambda^2U)\}^2$$
$$=(V+2\lambda W+\lambda^2U)\{U+2\mu(W+\lambda U)+\mu^2(V+2\lambda W+\lambda^2U)\}.$$

Choosing λ, μ so that the co-fficients of z^2 vanish in both the factors on the right hand side,

$$C_1\lambda^2+2C\lambda+C_2=0,$$
$$C_1+2\mu(C+\lambda C_1)=0.$$

We thus reduce $W^2=UV$; to the findamental form (1).

We cannot have C, C_1 or C, C_2 both zero; nor $C_1C_2=C^2$ since the vertex (1,0,0) does not lie on the curve; hence λ, μ can always be found to effect the reduction.

Consider the form $W^2=UV$, this is the envelope of the system of conics
$$\lambda^2 U+2\lambda W+V=0. \quad \ldots \quad \ldots \quad \ldots (2)$$

The pole of $z=0$ with respect to (2) is given by
$$\lambda^2(H_1y+G_1z)+2\lambda(Hy+Gz)+H_2y+G_2z=0,$$
$$\lambda^2(H_1x+F_1z)+2\lambda(Hx+Fz)+H_2x+F_2z=0,$$
these being the polars of the points (1, 0, 0) and (0, 1,0) respectively. Writing these
$$\frac{x}{\lambda^2F_1+2\lambda F+F_2}=\frac{y}{\lambda^2G_1+2\lambda G+G_2}=\frac{z}{\lambda^2H_1+2\lambda H+H_2}$$
the elimination of λ must give a conic as the locus of this pole. For, any straight line $lx+my+nz=0$ meets the locus in the two points given by the values of λ satisfying the equation
$$l(\lambda^2F_1+2\lambda F+F_2)+m(\lambda^2G_1+2\lambda G+G_2)-n(\lambda^2H_1+2\lambda H+H_2)=0.$$

We may thus state the following general theorem.

The general binodal quartic is the envelope of a conic belonging to the net U, V, W of three conics through two fixed points (and not all passing through a third point) such that the pole of the line joining these two points with respect to the variable conic moves on a fixed conic.

There are four such nets of conics whereby the binodal quartic may be generated.

If the three conics all pass through a third point the corresponding quartic is trinodal having nodes at the three fixed points of intersection.

There are only four such nets; for, we have seen that $W^2=UV$ is identical with
$$\{W+\lambda U+\mu(V+2\lambda W+\lambda^2U)\}^2$$
$$=(V+2\lambda W+\mu^2U)\{U+2\mu(W+\lambda U)+\mu^2(V+2\lambda W+\lambda^2U)\}$$
and the net of conics U, V, W is precisely identical with the net
$$k_1U+k_2V+k_3W,\ k_1'U+k_2'V+k_3'W,\ k_1''U+k_2''V+k_3''W.$$

3. *The four conics, loci of the poles of the variable conics of the nets are inscribed in the quadrilateral formed by the four tangents to the binodal at its nodes.*

Take the fixed conic, locus of the poles, as
$$ax^2+by^2+cz^2+2hxy+2gxz+2fyz=0, \quad \ldots \quad \ldots (3)$$
and the quartic to be the envelope of
$$\lambda(Lyz+Mzx+Nxy+Pz^2)+\mu(L'yz+M'zx+N'xy+P'z^2)$$
$$+\nu(L''yz+M''zx+N''xy+P''z^2)=0,$$
where the pole of this conic with respect to $z=0$ lies on the fixed conic
$$(a, b, c, f, g, h)\ (x, y\ z)^2=0.$$

The pole of $z=0$ is given by

$$\lambda(Mz+Ny)+\mu(M'z+N'y)+\nu(M''z+N''y)=0,$$
$$\lambda(Lz+Nx)+\mu(L'z+N'x)+\nu(L''z+N''x)=0,$$

this being the intersection of the polars of (0,1,0) and (1,0,0) :

Or $x:y:z=\lambda L+\mu L'+\nu L'' \quad \lambda M+\mu M'+\nu M'' :-(\lambda N+\mu N'+\nu N'')$.

Thus we require the envelope of

$$(\lambda L+\mu L'+\nu L'')yz+(\lambda M+\mu M'+\nu M'')zx+(\lambda N+\mu N'+\nu N'')xy$$
$$+(\lambda P+\mu P'+\nu P'')z^2=0,$$

subject to $\qquad (a, b. c, f, g, h)(L^x, M^x, -N^x)^2=0,$

writing $\lambda L+\mu L'+\nu L''=L^x, \lambda M+\mu M'+\nu M''=M^x, \lambda N+\mu N'+\nu N''=N^x$.

In other words, we require the envelope of

$$L^x yz+M^x zx+N^x xy=z^2\begin{vmatrix} 0, & P, & P', & P'' \\ L^x, & L, & L', & L'' \\ M^x, & M, & M', & M'' \\ N^x, & N, & N', & N'' \end{vmatrix} \div \begin{vmatrix} L, & L', & L'' \\ M, & M', & M'' \\ N, & N', & N'' \end{vmatrix}$$

$$=(\alpha L^x+\beta M^x+\gamma N^x)z^2$$

subject to $\qquad (a, b, c, f, g, h)\ (L^x, M^x, -N^x)^2=0.$

This envelope is

$$(A, B, C, F, G, H)\ (yz-\alpha z^2, zx-\beta z^2, -xy+\gamma z^2)=0,$$

where A, B, C, F, G, H are the usual minors in the discriminant of

$$(a, b, c, f, g, h)\ (xyz)^2=0.$$

Thus the quartic is

$$A(yz-\alpha z^2)^2+B(zx-\beta z^2)^2+C(xy-\gamma z^2)^2+2H(yz-\alpha z^2)\ (zx-\beta z^2)$$
$$-2F(zx-\beta z^2)(xy-\gamma z^2)-2G(yz-\alpha z^2)(xy-\gamma z^2)=0.$$

The tangents at the nodes $y=0$, $z=0$ and $x=0$, $z=0$ are given by

$$Bz^2+Cy^2-2Fyz=0,$$
$$Az^2+Cx^2-2Gxz=0.$$

Now the condition $\lambda x+\mu y+\nu z=0$ may touch the conic (3) is

$$(A, B, C, F, G, H)\ (\lambda, \mu, \nu)^2=0.$$

If therefore $\lambda x+\nu z=0$ is a tangent from $z=0$, $x=0$, we get for the pair of tangents from (0, 1, 0) the equations

$$(A, B, C, F, G, H)\ (z, o, -x)^2=0,$$

or $$Az^2+Cx^2-2Gxz=0.$$

Similarly the tangents from (1,0,0) are given by

$$Bz^2+Cy^2-2Fyz=0.$$

Thus the conic, locus of the poles, is inscribed in the quadrilateral formed by the four tangents at the nodes and thus the four directing

conics are all inscribed in the same quadrilateral. When the nodes are at the circular points at infinity, the conics U, V, W of § 2 are circles ; and a net of circles, say $\lambda C_1+\mu C_2+\nu C_3=0$, where C_1, C_2, C_3 are fixed circles, can be expressed otherwise. There is a single circle C that cuts orthogonally C_1, C_2, C_3 and also every circle of the form $\lambda C_1+\mu C_2+\nu C_3=0$, and conversely. Hence we have the theorem stated in § 1 :

A bicircular quartic can in general be generated in four different ways as the envelope of a circle that cuts a fixed circle orthogonally and whose centre moves on a fixed conic. The four fixed conics so obtained are confocal, their foci being the double foci of the bicircular quartic.

4. Consider a circular cubic and the line at infinity as the limit of a bicircular quartic. We obtain the theorem :

A circular cubic can be generated as the envelope of a circle which cuts a fixed circle orthogonally and whose centre moves on a fixed parabola.

This can be done in general in four ways and the parabolas are confocal.

Again, the trinodal quartic may be looked upon as the envelope of $\lambda yz+\mu zx+\nu xy=0$, subject to the condition

$$a\lambda^2+b^2\mu+c\nu^2+2h\lambda\mu+2g\lambda\nu+2f\mu\nu=0.$$

The trinodal is

$$Ay^2z^2+Bz^2x^2+Cx^2y^2+2Hxyz^2+2Gxy^2z+2Fx^2yz=0.$$

The six tangents at the nodes are

$$Ay^2+Bx^2+2Hxy=0,$$
$$Az^2+Cx^2+3Gxz=0,$$
$$Bz^2+Cy^2+2Fyz=0.$$

As in § (3) these six tangents touch the conic whose tangential equation is $(A, B, C, -F, -G, -H)(\lambda,\mu,\nu)^2=0$, and the point equation is therefore

$$(BC-F^2)x^2+\ldots 2(FG+CH)xy+\ldots=0.$$

This proof is the same virtually as that in Basset, § 192.

5. The general theorems obtained give no indication respecting the reality of the focal conic or fixed circles. We proceed to determine the reality of the four methods of generation and the corresponding classification of bicircular quartics.

The general real bicircular quartic is given by

$$(x^2+y^2)^2+4(x^2+y^2)(lx+my)+ax^2+2hxy+by^2+2gx+2fy+c=0,$$

which can be written

$$(x^2+y^2+2lx+2my+l^2+m^2)^2=a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'.$$

Changing the origin, this becomes

$$(x^2+y^2)^2=ax^2+2hxy+by^2+2gx+2fy+c, \text{ say.}$$

If the discriminant of the right hand side is not zero, consider the equation

$$(x^2+y^2+\delta)^2=(a+2\delta)x^2+2hxy+(b+2\delta)y^2+2gx+2fy+c+\delta^2.$$

The discriminant of the right hand side will vanish, if

$$(a+2\delta)(b+2\delta)(c+\delta^2)+2fgh-(a+2\delta)f^2-(b+2\delta)g^2-(c+\delta^2)h^2=0.$$

The roots of the equation $(a+2\delta)(b+2\delta)-h^2=0$ are real and they make the remaining terms $(a+2\delta)f^2+(b+2\delta)g^2-2fgh$ a perfect square. Just as in the discussion of the discriminating cubic in Solid Geometry, these squares are one positive and one negative. Therefore the equation in δ has at least one real root. Thus by transformation of axes we can reduce the bicircular quartic to the standard form

$$(x^2+y^2+2gx+2fy+c)^2=ax^2+by^2;$$

where, in general, neither a nor b is zero. If one of a, b is zero, the bicircular degenerates to a pair of circles.

If, however, $h=0$ and $a=b$, the equation

$$(x^2+y^2+\delta)^2=(a+2\delta)(x^2+y^2)+2gx+2fy+c+\delta^2$$

which represents a cartesian, will have its right hand side a point circle if $(a+2\delta)\{(a+2\delta)(c+\delta^2)-f^2-g^2\}=0$, and the cubic in δ has then one real root.

Thus every real bicircular quartic, including cartesians, can be reduced to the form

$$(x^2+y^2-c)^2=a(x-\alpha)^2+b(y-\beta)^2.$$

6. Consider now the bicircular quartic generated by a circle cutting $(x-\alpha)^2+(y-\beta)^2=\gamma^2$ orthogonally and whose centre describes the conic $\frac{x^2}{a}+\frac{y^2}{b}=1$. Write $a_1=\sqrt{a}$, $b_1=\sqrt{b}$.

The circle $(x-a_1\cos\theta)^2+(y-b_1\sin\theta)=r^2$ cuts the given circle orthogonally, if

$$(\alpha-a_1\cos\theta)^2+(\beta-b_1\sin\theta)^2=\gamma^2+r^2.$$

Subtracting, we require therefore to find the envelope of the system of circles

$$x^2+y^2-2(x-\alpha)a_1\cos\theta-2(y-\beta)b_1\sin\theta-\alpha^2-\beta^2+\gamma^2=0 \quad \dots (1)$$

The envelope is obtained by eliminating θ between this equation and

$$(x-\alpha)a_1\sin\theta-(y-\beta)b_1\cos\theta=0. \quad \dots (2)$$

The envelope is therefore

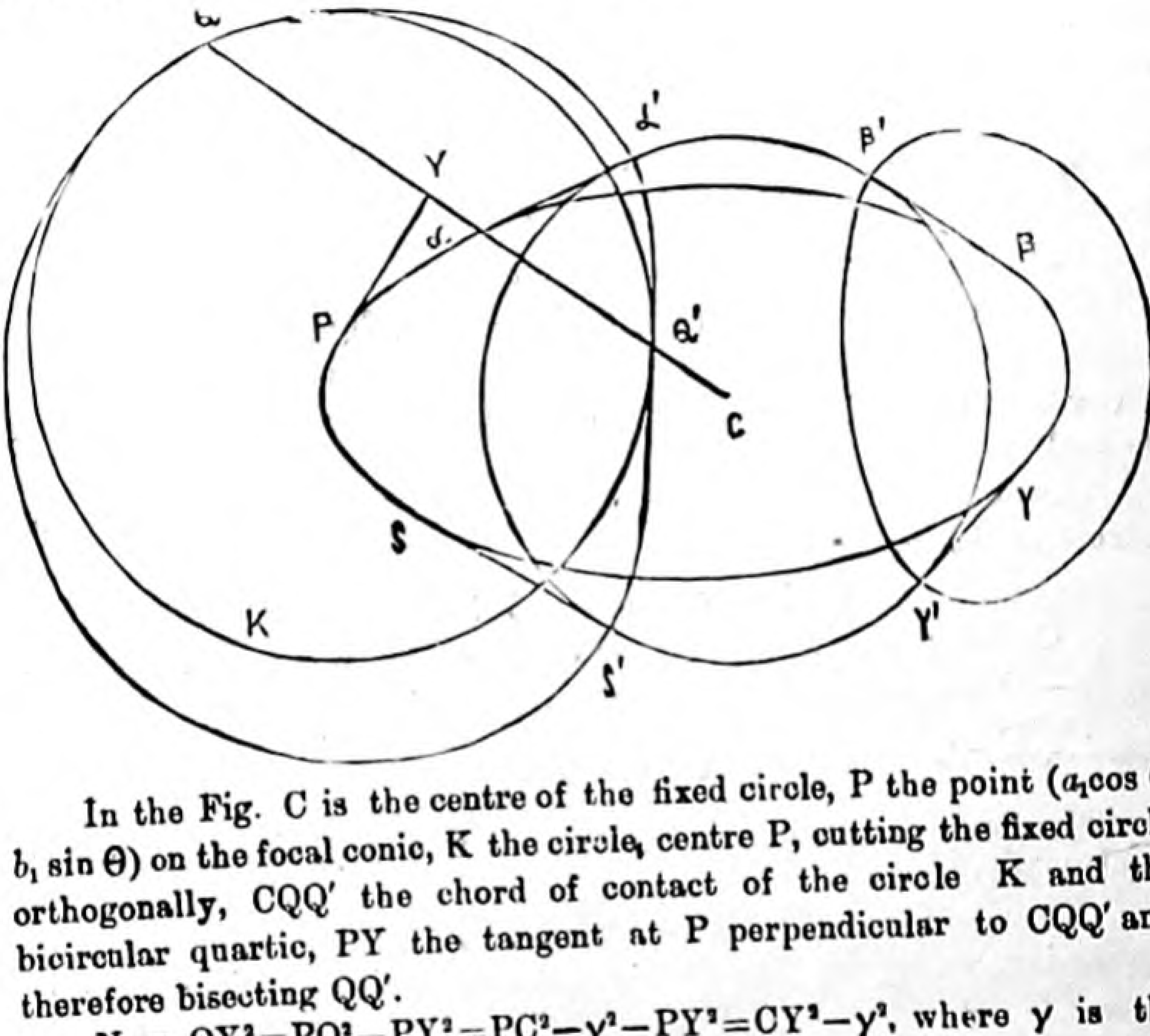
$$(x^2+y^2-\alpha^2-\beta^2+\gamma^2)^2=4a(x-\alpha)^2+4b(y-\beta)^2. \quad \dots (3)$$

Hence we derive the theorem:

The bicircular quartic $(x^2+y^2-c)^2=4a(x-\alpha)^2+4b(y-\beta)^2$ *has* $\frac{x^2}{a}+\frac{y^2}{b}=1$ *for a focal conic, and* $(x-\alpha)^2+(y-\beta)^2=\alpha^2+\beta^2-c$, *for the corresponding fixed circle.*

The circle is real only if $\alpha^2+\beta^2>c$.

Some interesting geometrical theorems and another method of generating the bicircular can be now obtained. The chord of contact of the generating circle (1) with the bicircular (3) is the straight line (2) which passes through the centre (α, β) of the fixed circle and is perpendicular to $\frac{x}{a_1}\cos\theta+\frac{y}{b_1}\sin\theta=1$ which is the tangent at the point $(a_1\cos\theta, b_1 \sin\theta)$ to the focal ellipse.



In the Fig. C is the centre of the fixed circle, P the point ($a_1\cos\theta$ $b_1\sin\theta$) on the focal conic, K the circle, centre P, cutting the fixed circle orthogonally, CQQ′ the chord of contact of the circle K and the bicircular quartic, PY the tangent at P perpendicular to CQQ′ and therefore bisecting QQ′.

Now $QY^2=PQ^2-PY^2=PC^2-\gamma^2-PY^2=CY^2-\gamma^2$, where γ is the radius of the fixed circle.

We have therefore a second method of generating a general bicircular quartic:

From a fixed point C draw the perpendicular CY on the tangent to a conic, the locus of points Q on CY such that CY^2-QY^2 is constant, is a bicircular quartic.

If CY^2-QY^2 is positive the corresponding fixed circle is real, but if CY^2-QY^2 is negative the corresponding fixed circle is imaginary.

The fixed circle we have hitherto spoken of is also called the *circle of inversion*; for, if the bicircular quartic is inverted with respect to it

the inverse figure is the same bicircular quartic. This is obvious since $CQ \cdot CQ' = \gamma^2$.

There are four points with respect to which a bicircular quartic can be inverted into itself.

The normals, at Q, Q' to the bicircular intersect on the focal conic at P. Basset, §206.

7. Consider the bicircular

$$(x^2+y^2-c)^2 = 4a(x-\alpha)^2+4b(y-\beta)^2$$

Identifying this with

$$(x^2+y^2-c') = 4(a+\lambda)(x-\alpha')^2+4(b+\lambda)(y-\beta')^2$$

we have
$$2c'+4\lambda = 2c$$
$$c'^2-4a\alpha'^2-4b\beta'^2-4\lambda(\alpha'^2+\beta'^2) = c^2-4a\alpha^2-4b\beta^2$$
$$a\alpha'+\lambda\alpha' = a\alpha$$
$$b\beta'+\lambda\beta' = b\beta.$$

Hence
$$2(c'-c) = -4\lambda$$
$$a(\alpha'-\alpha) = -\lambda\alpha'$$
$$b(\beta'-\beta) = -\lambda\beta'$$
$$c'^2-c^2-4a(\alpha'^2-\alpha^2)-4b(\beta'^2-\beta^2)-4\lambda(\alpha'^2+\beta'^2) = 0.$$

By use of the first three equations, the last becomes

$$c'+c-2\alpha'(\alpha+\alpha')-2\beta'(\beta+\beta')+2\alpha'^2+2\beta'^2 = 0,$$

or finally
$$c = \lambda+\frac{a\alpha^2}{a+\lambda}+\frac{b\beta^2}{b+\lambda} \quad \ldots \quad \ldots \text{(1)}$$

$$\alpha' = \frac{a\alpha}{a+\lambda},\ \beta' = \frac{b\beta}{b+\lambda},\ c' = c-2\lambda.$$

The cubic in λ gives the three other methods of generating the bicircular by means of the focal conics $\frac{x^2}{a+\lambda}+\frac{y^2}{b+\lambda} = 1$, and the circles $\left(x-\frac{a\alpha}{a+\lambda}\right)^2+\left(y-\frac{b\beta}{b+\lambda}\right)^2 = \frac{a^2\alpha^2}{(a+\lambda)^2}+\frac{b^2\beta^2}{(b+\lambda)^2}-c+2\lambda$, where λ is a root of (1).

These fixed circles are mutually orthogonal.

For, consider the two circles

$$x^2+y^2-2\alpha x-2\beta y+c = 0,$$
$$x^2+y^2-\frac{2a\alpha}{a+\lambda}x-\frac{2b\beta}{b+\lambda}y+c-2\lambda = 0;$$

these are orthogonal, if

$$\frac{a\alpha^2}{a+\lambda}+\frac{b\beta^2}{b+\lambda}-c+\lambda = 0, \text{ which is condition (1).}$$

8. The bitangents of the bicircular

$$(x^2+y^2-c)^2=a(x-\alpha)^2+b(y-\beta)^2$$

are given by
$$a(x-\alpha)^2+b(y-\beta)^2=0,$$
and
$$(a+\lambda)(x-\alpha')^2+(b+\lambda)(y-\beta')^2=0,$$
where λ, α', β' are determined as in § 5.

The points of contact lie on the four concentric circles

$$x^2+y^2-c=0,\ x^2+y^2-c+2\lambda=0,$$

where λ is a root of (1). Also

$$2\{a(x-\alpha)^2+b(y-\beta)^2\}+\lambda(x^2+y^2-c),$$
$$\equiv 2\{a+\lambda)(x-\alpha')^2+(b+\lambda)(y-\beta')^2\}-\lambda(x^2+y^2-c+2\lambda)\}$$

in virtue of the relations of § 5.

Thus the eight points of contact of the two pairs of bitangents lie on the conic

$$2\{a(x-\alpha)^2+b(y+\beta)^2\}+\lambda(x^2+y^2-c)=0.$$

A similar theorem is true for the general quartic.

If λ_1, λ_2, λ_3 are the roots of the equation,

$$\frac{a\alpha^2}{a+\lambda}+\frac{b\beta^2}{b+\lambda}=c-\lambda,$$

we obtain

$$\frac{a\alpha^2}{(a+\lambda_1)(a+\lambda_2)}+\frac{b\beta^2}{(b+\lambda_2)(b+\lambda_2)}=\frac{1}{4}$$
$$\frac{a\alpha^2}{(a+\lambda_1)(a+\lambda_2)(a+\lambda_3)}+\frac{b\beta^2}{(b+\lambda_1)(b+\lambda^2)(b+\lambda_3)}=0, \qquad \text{... (1)}$$

Every real bicircular quartic has two real bitangents.

For, if a, b have opposite signs the pair of straight lines $a(x-\alpha)^2+b(y-\beta)^2=0$ is real.

If a, b have the same sign then, if $\lambda_1\lambda_2\lambda_3$ are all real, from (1) at least one pair of values $a+\lambda$, $b+\lambda$ must have opposite signs; if, however, λ_1, λ_2, are conjugate imagineries, then both $(a+\lambda_1)(a+\lambda_2)$ and $(b+\lambda_1)(b+\lambda_2)$ are positive and thus from (1) $a+\lambda_3$, $b+\lambda_{33}$ have opposite signs.

A real bicircular quartic has either two real or six real bitangents, but cannot have four or eight.

For, from (1) when $\lambda_1,\lambda_2,\lambda_3$ are all real the signs of the pairs a, b; $a+\lambda_1$, $b+\lambda_1$; $a+\lambda_2$, $b+\lambda_2$, $a+\lambda_3$, $b+\lambda_3$ must be either three alike and one different, or one alike and three different; when two of the values of λ are imaginary, say λ_1, λ_2, the pairs a, b; $(a+\lambda_3)$, $(b+\lambda_3)$; must be one alike and one different in signs.

9. I proceed now to discuss the generation of the bicircular quartic by means of real focal conics and real circles of inversion.

Since there is always one pair of imaginary and therefore conjugate imaginary bitangents, every real bicircular quartic can be written

$$(x^2+y^2-c)^2=4a^2(x-\alpha)^2+4b^2(y-\beta)^2$$

where all the letters denote real quantities.

Corresponding to the equivalent form

$$(x^2+y^2-c+2\lambda)^2=4(a^2+\lambda)(x-\alpha')^2+4(b^2+\lambda)(y-\beta')^2$$

we have

$$\alpha'=\frac{a^2\alpha}{a^2+\lambda},\ \beta'=\frac{b^2\beta}{b^2+\lambda},\ c=\lambda+\frac{a^2\alpha^2}{a^2+\lambda}+\frac{b^2\beta^2}{b^2+\lambda},$$

and the radius of the corresponding circle of inversion is

$$\begin{aligned}\gamma'^2&=\alpha'^2+\beta'^2-c+2\lambda\\ &=\frac{a^4\alpha^2}{(a^2+\lambda)^2}+\frac{b^4\beta^2}{(b^2+\lambda)^2}-\frac{a^2\alpha^2}{a^2+\lambda}-\frac{b^2\beta^2}{b^2+\lambda}+\lambda\\ &=\lambda\left\{1-\frac{a^2\alpha^2}{(a^2+\lambda)^2}-\frac{b^2\beta^2}{(b^2+\lambda)^2}\right\}=\lambda\frac{\partial c}{\partial\lambda}.\end{aligned}$$

Consider now the curve

$$y=x+\frac{a^2\alpha^2}{a^2+x}+\frac{b^2\beta^2}{b^2+x}.$$

It has asymptotes $y=x$, $x+a^2=0$, $x+b^2=0$.

The turning points are given by the equation

$$1-\frac{a^2\alpha^2}{(a^2+x)^2}-\frac{b^2\beta^2}{(b^2+x)^2}=0,$$

or $$(a^2+x)^2(b^2+x)^2-a^2\alpha^2(b^2+x)^2-b^2\beta^2(a^2+x)^2=0.$$

The signs of the function on the left hand side are

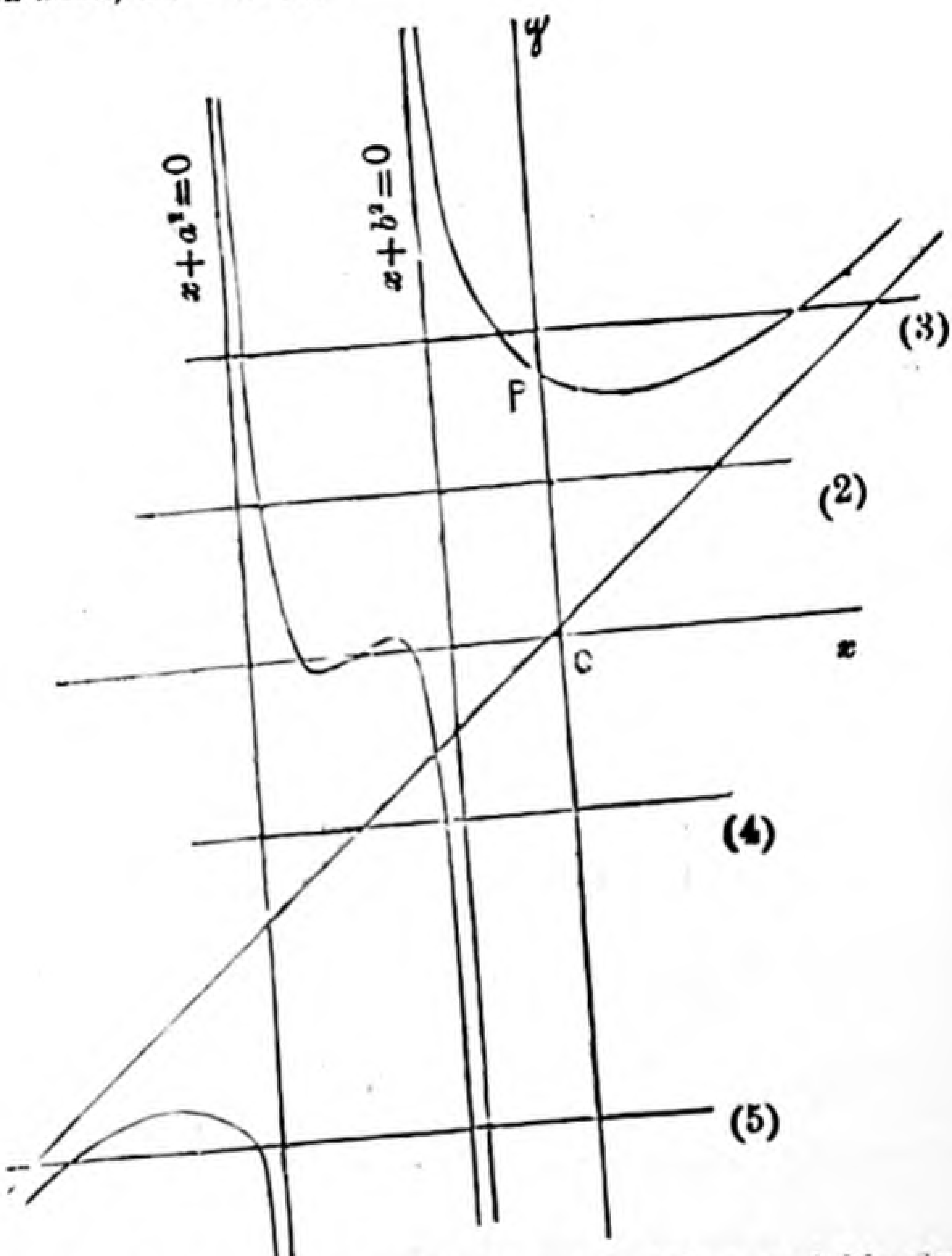
for $x=+\infty$, $x=-b^2$, $x=-a^2$, $x=-\infty$.

$+$, $-$, $-$, $+$.

There is therefore one turning point on each of the branches having $y=x$ for asymptote and from geometrical considerations there cannot be three turning points on either of these branches.

Thus the most general case occurs when this equation has four real roots. The corresponding curve is represented in the Fig.

When $x=0$, $OP=\alpha^2+\beta^2$.



Consider the intersections of the curve with a variable straight line $y=c$, then at the points of intersection the following cases arise.

I. A straight line close to the axis of x intersects the branch of the curve lying between the parallel asymptotes in three real points.

Let the values of λ from left to right be λ_1, λ_2, λ^3.

II. A straight line as (2) or (4) in the figure intersects the curve in only one real point.

III. A straight line below P with three real intersections λ_1, λ_2, λ^3 from left to right.

(1) The turning point on the branch is to the right of P.

(2) The turning point on the branch is to the left of P.

IV. A straight line above P as (3) in the figure with three real intersections λ_1, λ_2, λ_3 from left to right.

V. A straight line as (5). At the first two points of intersection from the left, both $a^2+\lambda$ and $b^2+\lambda$ are negative. The bicircular is therefore purely imaginary.

The following table gives the signs of $a^2+\lambda$, $b^2+\lambda$, $\gamma'^2=\lambda\dfrac{\partial c}{\partial\lambda}=x\dfrac{dy}{dx}$ for the point of intersection, c, and $\gamma^2=\alpha^2+\beta^2-c$.

Case.	$a^2+\lambda$	$b^2+\lambda$	γ'^2	c	γ^2
I λ_1	+	−	+	$\pm<\alpha^2+\beta^2$	+
„ λ_2	+	−	−	„	+
„ λ_3	+	−	+	„	+
II	+	−	+	$\pm<\alpha^2+\beta^2$	+
III (1) λ_1	+	−	+	$+<\alpha^2+\beta^2$	+
„ λ_2	+	+	−	„	+
„ λ_3	+	+	+	„	+
III (2) λ_1	+	−	+	„	+
„ λ_2	+	+	+	„	+
„ λ_3	+	+	−	„	+
IV λ_1	+	−	+	$+>\alpha^2+\beta^2$	−
„ λ_4	+	+	+	„	−
„ λ_3	+	+	+	„	−

An inspection of the table leads to the following conclusions:

1. *When the equation in λ has only one real root, the bicircular quartic can be generated in two ways only. One by a real focal ellipse and real circle of inversion, the other by a real focal hyperbola and real circle of inversion. There are only two real bitangents.*

2. *When the equation in λ has three real roots, all the focal conics are real, but only three of the circles of inversion are real.* (1) *The bicircular*

can be generated from two real focal ellipses and corresponding real circles of inversion, or from one real focal hyperbola and corresponding real circle of inversion. There are only two real bitangents. (2) *The bicircular can be generated from one real focal ellipse and corresponding real circle of inversion or from two real focal hyperbolas and corresponding real circle of inversion. There are six real bitangents.*

Considering the second method of generation in § 6, where $CY^2-QY^2=y^2$: we see that in (1) the bicircular can be generated from either an ellipse or a hyperbola; in (2) the bicircular can be generated in four real ways one of which is always from an ellipse and another always from a hyperbola.

10. It may be of interest to consider a numerical example and we select a case of type IV.

Consider the bicircular given by

$$\left(x^2+y^2-\frac{5}{2}\right)^2=12\left(x-\frac{2}{\sqrt{3}}\right)^2+8\left(y-\frac{\sqrt{3}}{2}\right)^2.$$

The equation in λ is

$$\frac{5}{2}=\lambda+\frac{4}{3+\lambda}+\frac{3}{4+2\lambda}.$$

This has roots $\pm 1, -\frac{5}{2}$.

For $\lambda=1$, $\alpha'=\frac{\sqrt{3}}{2}$, $\beta'=\sqrt{3}$, $c'=\frac{1}{2}$,

„ $\lambda=-1$, $\alpha'=\sqrt{3}$, $\beta'=\sqrt{3}$, $c'=\frac{9}{2}$,

„ $\lambda=-\frac{5}{2}$, $\alpha'=4\sqrt{3}$, $\beta'=-2\sqrt{3}$, $c'=\frac{15}{2}$

The equivalent forms of the bicircular quartic are, therefore,

$$\left(x^2+y^2-\frac{1}{2}\right)^2=16\left(x-\frac{\sqrt{3}}{2}\right)^2+12\left(y-\frac{1}{\sqrt{3}}\right)^2$$

$$\left(x^2+y^2-\frac{9}{2}\right)^2=8(x-\sqrt{3})^2+4(y-\sqrt{3})^2$$

$$\left(x^2+y^2-\frac{15}{2}\right)^2=2(x-4\sqrt{3})^2-2(y+2\sqrt{3})^2.$$

All these reduce to

$$(x^2+y^2)^2=17\,x^2+13\,y^2-16\sqrt{3}\,x-8\,\sqrt{3}\,y+15\,\tfrac{3}{4}.$$

The corresponding methods of generation are by means of the respective focal conics and fixed circles of inversion given by:

$$\frac{x^2}{3}+\frac{y^2}{2}=1\,;\; x^2+y^2-\frac{4}{\sqrt{3}}x-\sqrt{3}\,y+\frac{5}{2}=0,$$

$$\frac{x^2}{4}+\frac{y^2}{3}=1\,;\; x^2+y^2-\sqrt{3}\,x-\frac{2}{\sqrt{3}}y+\frac{1}{2}=0,$$

$$\frac{x^2}{2}+\frac{y^2}{1}=1\,;\; x^2+y^2-2\sqrt{3}\,x-2\sqrt{3}\,y+\frac{9}{2}=0,$$

$$2x^2-2y^2=1\,;\; x^2+y^2-8\sqrt{3}\,x+4\sqrt{3}\,y-\frac{15}{2}=0.$$

The first circle only is imaginary; the only real bitangents are

$$x-4\sqrt{3}=\pm(y+2\sqrt{3}).$$

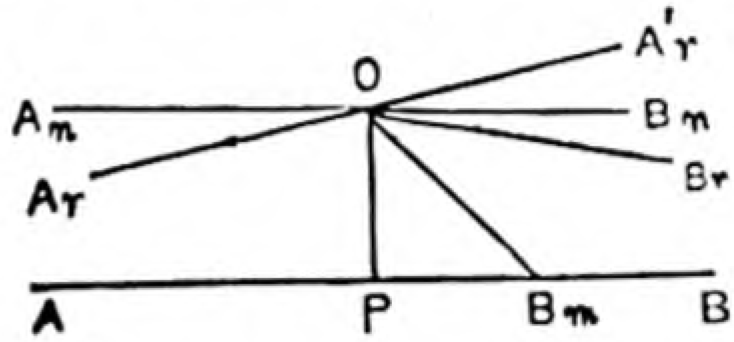
(To be continued.)

SHORT NOTES.

Parallel Straight Lines.

[The following is the abstract of a paper read by the writer before the Mathematical Association of the Presidency College, Madras, in January last. Reference: Coolidge's *Non-Euclidean Geometry*.]

Let O be a point and AB a line. Draw OP perpendicular to AB.



Consider the rays OB_n lying in one of the *half*-planes bounded by OP. Then the following alternatives are possible:

(A) *None of the rays OB_n meets PB,*

(R) *All the rays OB_n meet PB,*

(EL) *Some of the rays OB_n meet PB, and some do not meet it.*

Case (A) can be discarded. For, it is tantamount to saying that we cannot connect O and a point in PB by a line.

Case (R) applies to finite space, as can be easily seen. Draw OA_n, OB_n on different sides of OP, making equal angles with it. Then by congruence we prove that OA_n, OB_n meet AB at equal distance from P. Take $P\hat{O}A_n=P\hat{O}B=90^\circ$ and we deduce that the perpendiculars to the same line OP meet at equal distances on opposite sides of it. Calling this distance Δ, we easily prove that Δ is independent of the length of OP. If we further agree that two lines cannot have more than one point in common, then the two points of intersection above are identical, and

"*All straight lines are of constant length* 2Δ."

This is the assumption of *Riemann* and his geometry is therefore one of finite space.

Case (EL). Let OB_m be any ray which meets PB, and let OB_n be any ray which does not meet it. Then, evidently, OB_n cannot lie in the angle POB_m. Therefore

$$P\hat{O}B_n > POB_m,$$

and, by Dedekind's principle, we can form a cut determining uniquely a ray OB_r, which separates the rays which *meet* PB from the rays which *do not meet* it, and which itself may be assigned to either class.

Def: The ray OB_r thus determined is defined to be *parallel* to the ray PB at the point O, and the angle POB is the *angle of parallelism* for the distance OP.

From symmetry there will be a ray OA_r in the other *half*-plane parallel to the ray PA, so that

$$P\hat{O}A_r = P\hat{O}B_r = \text{angle of parallelism.}$$

Now, either

(E) *The rays OA_r, OB_r are the same line,*

or (L) *They are different lines.*

(1) The former case gives one and only one line parallel to a given line through a given point, and is the ordinary *Euclidean* assumption. The angle of parallelism here is a right angle.

(2) The latter gives two lines through any point parallel to a given line, and is the *Lobatchewskian* assumption. The angle of parallelism in this case is less than a right angle.

If A_rO is produced to A_r', then rays which lie within the angle. B_rOA_r' lie also within the regetically opposite angle, and therefore do not meet AB at either end.

Def: Such lines, as the above, which neither intersect nor are parallel are defined to be *skew lines.*

The following theorems relate to (L) and (E):

THEOREM I. *Parallel rays meet at infinity.*

This is at once obvious. For, the parallel ray through O belongs both to the set which meets PB and to the set which does not meet it (at a finite distance).

Cor. 1. In (E) the rays OA_r, OB_r, being the same can have only one point common with AB. Thus a line has only one point at infinity.

In (L) the rays OA_r, OB_r are different and therefore the rays PA, PB have distinct points at infinity.

[In (R) the points at infinity are imaginary.]

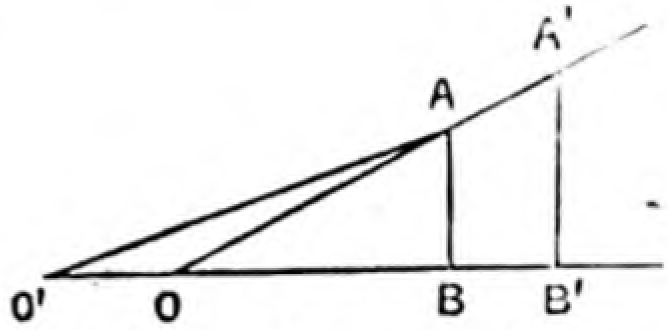
Cor. 2. If a ray be parallel to another ray at one point in it, it is so also at any other point in it. If a ray is parallel to another, the second is parallel to the first. Rays parallel to the same ray are parallel to one another.

These results follow by considering the properties of the point at infinity.

THEOREM II. (*a*) *The distance between two intersecting rays increases as the rays are produced and becomes ultimately infinite*; (*b*) *the distance between two parallel rays decreases in the direction of parallelism, and becomes ultimately zero*; and (*c*) *the distance between two skew lines becomes ultimately infinite both ways.*

(a) Let OA, OB be two intersecting rays. From points A, A′ on OA drop perpendiculars AB, A′B′ on OB. Let OB′>OB. Then we have to prove that A′B′>AB.

If not, suppose A′B′=AB. Apply OA′B′ to OAB so that A′B′ coincides with AB and B′O falls along BO. Then O falls without BO, on O′ say. And thus



$$A\hat{O}B > A\hat{O'}B \text{ or } A'\hat{O}B',$$

which is absurd.

If A′B′<AB, produce B′A′ to A″, so that A″B′=AB, then by the same procedure we reach a similar absurdity.

Hence A′B′>AB.

To prove that there is no superior limit to A′B′: suppose l to be such a limit. Choose OB′ so great that l is negligible in comparison with OB′. Then A′OB′ is negligible in comparison with OB′A′ or a right angle, which need not be true. Hence, there cannot be a limit l.

(b) Let OC, PB be two parallel rays, OP being ⊥ to PB. From a point O′ in OC draw O′P′ ⊥ to PB. Let QR be the ⊥r bisector of PP′. Then since QRO′ is an angle of parallelism in (L), it is less than a right angle.

$$\therefore Q\hat{R}O' < Q\hat{R}O.$$

By folding the quad. OPP′O′ along QR, it is obvious that O′ falls within OP.

$$\therefore O'P' < OP.$$

To shew that there is no inferior limit to O′P′; suppose l to be such a limit. Take PD sufficiently large in PB and draw DC ⊥ to OC,

CB$\perp$ to PB. Then by Th. I. OD can be made to coincide with OC in the limit. Choose PD so large that DC$<l$.

$$\therefore \qquad CB<CD<l.$$

In other words, there is no inferior limit to the approach of OC and PB.

(*c*) Let AB, CD be two skew lines. Draw AK parallel to CD. Then by (*a*) the distance between AB, AK is ultimately infinite, and by (*b*) that between AK and CD is ultimately zero. Also, since AB is skew to CD, AK lies between AB and CD. Hence the distance between AB, CD is ultimately infinite towards B, D.

Similarly, it is infinite towards the opposite side.

Cor. 1. The distance between two parallel straight lines increases infinitely in the direction opposite to that of parallelism.

Cor. 2. A rectangle cannot exist in (L). For, if OPP′O′ be a rectangle, by folding across QR, it is seen that O″ must coincide with O, since RO″P is otherwise greater than ROP. Thus, the opposite sides O′P′ and OP should be equal.

Also, from the congruence of OPQR and O′P′QR, it follows that OPQR is a rectangle. Therefore

$$OP=RQ=O'P'.$$

By repeating a similar construction in the case of the rectangles OPQR, RQP′O′, we find the opposite sides OO′, PP′ of the rectangle OPP′O′ preserve a *constant* distance between them.

We have seen that such lines do not exist in (L). Hence the result.

[A rectangle does not exist in (R) for the same reason.]

Theorem III. *Two skew lines have one and only one common perpendicular which is the shortest distance between them.*

For, since two skew lines diverge infinitely both ways the distance between them must be a minimum at least once. Also, it is readily seen that the common perpendicular is a minimum. Hence there must be at least one common perpendicular. There cannot be more than one; for, then a rectangle will exist.

Hence the theorem.

R. Vythynathaswamy.

Note on the Normal Section of an Enveloping Cylinder.

In the January (1914) number of the Edinburgh Mathematical Society's *Mathematical Notes*, Dr. R. J. T. Bell has an interesting paper on the Principal Axes of a Normal Section of an Enveloping Cylinder of

an Ellipsoid. His method of reasoning in the concluding part of the paper, however, does not seem to be quite convincing.

Having proved that

(1) Any radius vector of an ellipse is equal to the perpendicular from the centre to a tangent;

(2) The central perpendiculars to the tangent planes of an enveloping cylinder are equal to the semi-diameters of a certain section of the ellipsoid;

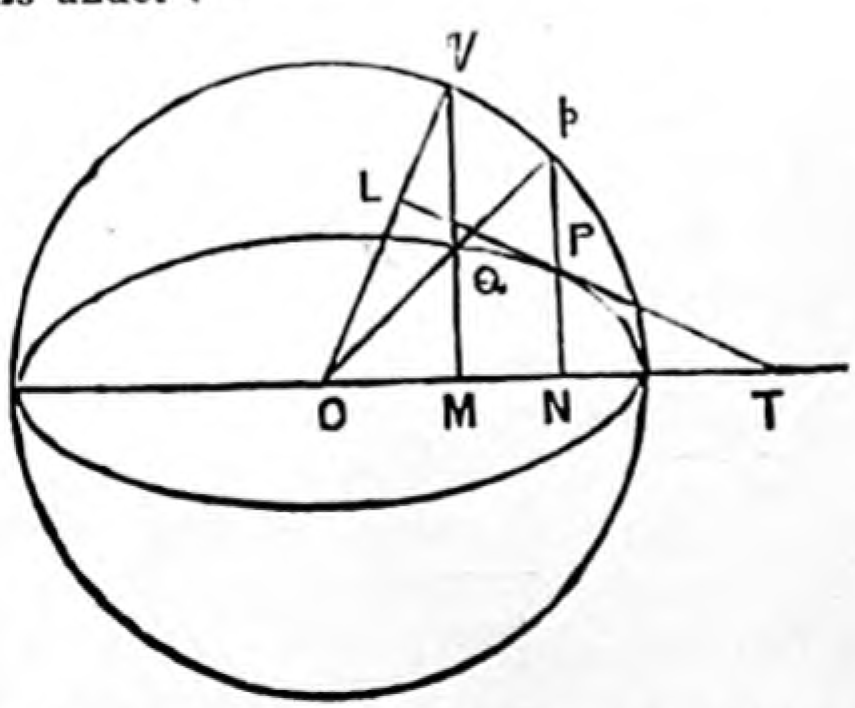
he concludes that the section of the ellipsoid must be equal to the normal section of the cylinder.

Now, for the equality of two ellipses it is not enough that any semi-diameter of one is equal to a semi-diameter of the other; but a further condition, such as the equality of angles between corresponding semi-diameter is necessary. For instance, in the ellipses (a, b) (a', b') any diameter of the first is equal to a diameter of the second, if $a' > a > b > b'$, though the ellipses are distinct. The additional condition required in the present case is easily supplied by the following further property of the ellipse:

The central perpendiculars on conjugate diameters are equal to two perpendicular semi-diameters of the ellipse and have the sum of the squares of their reciprocals constant.

From this we deduce that the sum of the squares of the reciprocals of the principal axes is the same in the two sections considered by Dr. Bell, whence the equality of the sections is established.

The properties referred to above admit of easy geometrical demonstration as under :—



(i) In the figure OL is the central perpendicular on the tangent at a point P of an ellipse, corresponding to p on the auxliary circle and Op

meet the ellipse at Q. To prove that OL meets the auxiliary circle at q which corresponds to Q.

We have

$$\frac{OM}{OQ}=\frac{ON}{OP}=\frac{OP}{OT}=\frac{OQ}{OT}=\frac{OM}{OL}, \text{ since } qLMT \text{ is cyclic.}$$

$$\therefore \qquad OQ=OL.$$

In other words, *the perpendicular on the tangent at P is equal to the semi-diameter of the ellipse in the direction Op.*

(ii) In the ellipsoid, let p (al, am, an) be the point of the auxiliary sphere corresponding to a point P (al, bm, cn) of the ellipsoid. Then if Op meet the ellipsoid at Q (al', bm', cn'), the corresponding point q (al', am', an') of the sphere is on the central perpendicular OL to the tangent plane at P, and OQ=OL.

For the proof, the steps are exactly the same as in (i) except that the point P in the figure is, in general, outside the plane of the paper.

(iii) The normal section of the enveloping cylinder whose generators are parallel to OQ is of area A′, such that

$$OQ\cdot A'=\pi\, abc. \text{ (Property of conjugate diameters.)}$$

Also, the central section A conjugate to OP is such that

$$OL.A=\pi\, abc.$$

Hence $$OQ\cdot A'=OL\cdot A.$$

Thus, the sections A and A′ are equal.

1*st March* 1914. M. T. NARANIENGAR.

The Face of the Sky for May and June 1914.

Sidereal time at 8 p.m.

	May.			*June.*	
D.	H.	M.		H.	M.
1	10	36	...	0	38
8	11	3	...	1	5
15	11	31	...	1	33
22	11	58	...	2	1
29	12	26	...	2	28

From this table the constellations visible during the evenings of May and June can be ascertained by a reference to the positions as given in a star-atlas.

The Sun

enters Cancer on June 22 at 0-25 P.M.

Phases of the Moon.

	May.				*June.*		
	D.	H.	M.		D.	H.	M.
First Quarter ...	3	11	59	...	1	7	33
Full Moon ...	10	3	1	...	8	10	48
Last Quarter ...	17	3	42	...	15	7	50
New Moon ...	25	8	5	...	23	9	3

Planets.

Mercury attains its greatest elongation (E) on June 1914. It is in conjunction with the Moon on May 26 at 6-37 A.M., and on June 25 and with Neptune on June 26.

Venus which is an evening star, is in conjunction with the Moon on May 27 and with Neptune on June 17.

Mars is in conjunction with the Moon on May 31 and June 28. It is visible in the evenings.

Jupiter is in quadrature on May 12; is stationary on June 12 when it begins to retrograde. It is in conjunction with the Moon on June 13.

Saturn is in conjunction with the Moon on June 23.

Uranus is in quadrature to the Sun on May 3 and stationary on May 17. It is in conjunction with the Moon on May 16.

Neptune is in conjunction with the Sun on May 29 and with the Moon on May 2.

V. RAMESAM.

SOLUTIONS.

Question 494.

(S. NARAYANA AIYAR, M.A. :—Prove that

$$\int_0^\infty \frac{\phi(a+ix)-\phi(a-ix)}{2\,ix}dx=\frac{\pi}{2}\left\{\phi(a)-\phi(\infty)\right\}.$$

Solution by R. Srinivasan, M.A., and R. Vythynathaswamy.

By Taylor's Theorem—

$$\phi(a+ix)-\phi(a-ix)=\sum\frac{a^n}{\lfloor n}\left\{\phi^n(ix)-\phi^n(-ix)\right\}.$$

$$\therefore \qquad 1=\sum\frac{a^n}{\lfloor n\cdot 2i}\int_0^\infty \frac{\phi^n(ix)-\phi^n(-ix)}{x}dx$$

$$=\sum\frac{a^n}{\lfloor n\cdot 2i}\left\{\phi^n(\infty)-\phi^n(o)\right\}\log\frac{ib^2}{-i},$$

$$=\frac{\log(-1)}{2i}\left\{\phi(a+\infty)-\phi(a+0)\right\},$$

by the theorem of Frullani, as modified by Elliot.

But $\log(-1)=\log e^{-i\pi}=-i\pi.$

Hence $$1=\frac{\pi}{2}\left\{\phi(a)-\phi(\infty)\right\}.$$

Question 497.

(T. P. TRIVEDI, M.A., L.L.B.,) :—Prove that

$$\int_0^\infty \frac{\sin rx}{x(1+x^2)^2}dx=\frac{\pi}{4}\left(2-e^{-r}(r+2)\right)$$

$$\int_0^\infty \frac{\sin^2 rx}{x^2(1+x^2)}dx=\frac{\pi}{4}\left(-1+e^{-2r}+2r.\right)$$

Solution (1) *by K. Narasinga Rao,* (2) *by V. K. Aravamudan and others.*

(1) We know that $\int_0^\infty \frac{\cos rx}{(a^2+x^2)}dx=\frac{\pi}{2a}e^{-ar}$ (Williamson, p. 151).

Differentiating with respect to a, we get

$$\int_0^\infty \frac{\cos rx}{(a^2+x^2)^2}dx=\frac{\pi e^{-ar}}{4a^3}(1+ar)$$

Integrating with respect to r

$$\int_0^\infty \frac{\sin rx}{x(a^2+x^2)^2}dx=\frac{\pi}{4a^3}\left[\int e^{-ar}\,dr+\int are^{-ar}\,dr\right]$$

$$=\frac{\pi}{4a^3}\left[-\frac{e^{-ar}}{a}-\frac{e^{-ar}}{a}-ar\frac{e^{-ar}}{a}\right]+\text{const.}$$

$$=-\frac{\pi}{4a^4}(ar+2)\,e^{-ar}+\text{const.}$$

Put $r=0$, and the constant is seen to be $\frac{\pi}{2a^4}$.

Thus $$\int_0^\infty \frac{\sin rx}{x(a^2+x^2)}dx=\frac{\pi}{4a^4}\left\{2-e^{-ar}(ar+2)\right\},$$

whence putting $a=1$, the first result is obtained.

Next, consider the identity

$$\frac{\sin^2 rx}{x^2(1+x^2)}=\left(\frac{1}{x^2}-\frac{1}{1+x^2}\right)\sin^2 rx=\frac{\sin^2 rx}{x^2}+\frac{\cos 2\,rx}{2(1+x^2)}-\frac{1}{2(1+x^2)}.$$

The second integral is thus equal to

$$\int_0^\infty \frac{\sin^2 rx}{x^2}dx+\tfrac{1}{2}\int_0^\infty \frac{\cos 2\,rx}{1+x^2}dx-\tfrac{1}{2}\int_0^\infty \frac{dx}{1+x^2}.$$

$$=\frac{\pi r}{2}+\frac{1}{2}\frac{\pi}{2}e^{-2r}-\frac{1}{2}\frac{\pi}{2}.\quad \text{(Gibson, p. 455.)}$$

$$=\frac{\pi}{4}(2r+e^{-r}-1).$$

(2) We know $\int_0^\infty \frac{\cos rx\,dx}{(1+x^2)^2}=\frac{\pi}{4}e^{-r}(1+r)$. (J. I. M. S. 1913, p. 230.)

Integrating with respect to r,

$$\int_0^\infty \frac{\sin rx}{x(1+x^2)^2}dx=-\frac{\pi}{4}e^{-r}(r+2)+C.$$

To find C, put $r=0$; we get $C=\frac{\pi}{2}$.

Hence the first integral $=\frac{\pi}{4}\left\{2-e^{-r}(r+2)\right\}$.

Again $\int_0^\infty \frac{\sin 2\,rx}{x(1+x^2)}dx=\frac{\pi}{2}(1-e^{-2r})$. (Gibson, p. 469).

Integrating with respect to r

$$\int_0^\infty \frac{\sin^2 rx}{x^2(1+x^2)}dx=\frac{\pi}{2}(r+\tfrac{1}{2}e^{-2r})+C.$$

Put $r=0$, then $C=-\frac{\pi}{4}$.

Hence the second integral$=\frac{\pi}{4}(-1+e^{-2r}+2r.)$

Question 498.

(R. N. Apte, M.A., F.R.A.S.) :—The parabola $y=ax^2+bx+c$ is drawn at random (where a, b, c, are real). Show that the probability of its cutting the x axis in imaginary points is $\frac{31}{72}-\frac{1}{12}\log 2$.

Solution (1) *by K. J. Sanjana M.A.*, (2) *by T. P. Trivedi M.A., L.L.B.*

(1) The $x-$axis being $y=o$, we get $ax^2+bx+c=0$; so that we have to find when the roots of this equation are imaginary. This requires $b^2<4ac$, which is impossible if a and c have opposite signs; and the chance that a and c have the same sign is $\frac{1}{2}$. Hence if, under this condition, the chance of $b^2<4ac$ is p, the probability required will be $\frac{1}{2}p$, when a and c are both taken positive.

The chance that one of the three a, b, c, is greater than the other two is $\frac{1}{3}$. The values of x depend only on the ratios of the coefficients, so that we can divide out by any one of these: if we divide out by the greatest, the other two are each made to lie between 0 and 1.

Let a be the greatest: the equation reduces to $x^2+bx+c=0$, with the condition $b^2<4c$. Thus $c>\frac{b^2}{4}$ which is less than unity; hence the limits of c are 1 and $\frac{b^2}{4}$, those of b being 1 and 0. The chance in this case$=\int_0^1 db\int_0^1 dc=\left\{b-\frac{b^3}{12}\right\}_0^1=\frac{11}{12}$. Similarly, if c is the greatest, the chance for imaginary roots is $\frac{11}{12}$.

Let b be the greatest, the equation now reduces to $ax^2+x+c=0$, with $1<4ac$. Thus $c>\frac{1}{4a}$, so that its limits are 1 and $\frac{1}{4a}$. The limits of a are no longer 1 and 0, for if $0<a<\frac{1}{4}$, $c>1$ which is impossible. Thus the chance in this case is

$$\int_{\frac{1}{4}}^{1} da\int_{\frac{1}{4a}}^{1} dc=\left\{a-\tfrac{1}{4}\log a\right\}_{\frac{1}{4}}^{1}=1-(\tfrac{1}{4}-\tfrac{1}{4}\log\tfrac{1}{4})=\tfrac{3}{4}-\tfrac{1}{2}\log 2.$$

Therefore finally, the required chance is

$$\tfrac{1}{2}\cdot\tfrac{1}{3}\left\{\frac{11}{12}+\frac{11}{12}+\frac{9}{12}-\tfrac{1}{2}\log 2\right\}=\frac{31}{72}-\frac{1}{12}\log 2.$$

(2) We have to find the probability that the roots of the equation $ax^2+bx+c=0$ are imaginary. We may suppose for the sake of convenience that a, b, c all lie between 1 and -1 as the equation may be divided out by any suitable number however large. The roots are imaginary if a and c have the same sign and $ac>\frac{b^2}{4}$.

The probability that a and c have the same sign is $\frac{1}{2}$. Let a and c be both positive and suppose p is the probability for imaginary roots ; then p is also the probability for imaginary roots when a and c are both negative. Therefore p is the probability for the roots being imaginary when a and c have the same sign whether positive or negative and the *absolute* probability for imaginary roots $=\frac{1}{2}p$. To find p we observe that ac being$>\frac{}{4}$, a may have any value from $\frac{b^2}{4}$ to 1 and then the values of c range from $\frac{b^2}{4a}$ to 1 ; also b lies between 1 and -1.

Hence
$$p=\frac{\int_{-1}^{1}\int_{\frac{1}{4}b^2}^{-1}\int_{\frac{1}{4}b^2}^{1} db\,da\,dc}{\int_{-1}^{1}\int_{0}^{1}\int_{0}^{1} db\,da\,dc};$$

the quantities a and c need only be taken between 1 and 0, since we confine our attention to positive values only.

$$\text{But}\int_{-1}^{1}\int_{\frac{1}{4}b^2}^{1}\int_{\frac{1}{4}b^2}^{1} db\,da\,dc=\int_{-1}^{1}\int_{\frac{1}{4}b^2}^{1}\left(1-\frac{b^2}{4}\right)db\,da$$

$$=\int_{-1}^{1}\left\{1-\frac{b^2}{4}+\frac{b^2}{4}\log\frac{b^2}{4}\right\}db$$

$$=2\int_{0}^{1}\left\{1-\frac{b^2}{4}+\frac{b^2}{2}\log\frac{b}{2}\right\}db=\frac{31}{18}+\frac{1}{3}\log\tfrac{1}{2};$$

and
$$\int_{-1}^{1}\int_{0}^{1}\int_{0}^{1} db\,da\,dc=2.$$

Thus $p=\frac{31}{36}+\frac{1}{6}\log\frac{1}{2}$, and the absolute probability is $\frac{1}{2}p=\frac{31}{72}-\frac{1}{12}\log 2$.

Question 499.

(K. J. SANJANA) :—A chord AKB of a curve is drawn perpendicular to the normal PK at any point P of the curve, and M is its mid-point; prove that when PK is infinitesimal the ratio of the radius of curvature at P to the radius of curvature at the corresponding point of the evolute is PK : 3 KM. (Suggested by Question 470.)

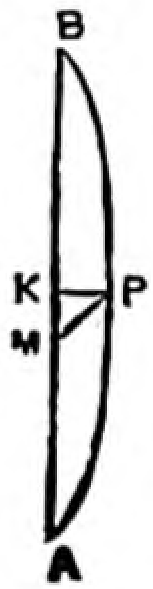
Solutions (1) *by T. P. Trivedi, M.A. L.L.B*; (2) *by T.S. Krishna Rao.*

(1) Let ρ be the radius of curvature at P on the curve and ρ' the the radius of curvature at the corresponding point of the evolute.

By Question 470 (J.I.M.S.) $\dfrac{\rho'}{\rho}=\dfrac{3y_1 y_2^2-y_3(1+y_1^2)}{y_2^2}$.

Also if the angle KPM be θ; then

$\tan\theta=\left\{ y_1-\dfrac{(1+y_1^2)y_3}{3y_2^2}\right\}$. [Edward's *Diff. Calc.* Ex. 38, page. 110.]



$$\therefore \qquad \frac{KM}{PK}=\frac{3y_1 y_2^2-y_3(1+y_1^2)}{3y_2^2}=\frac{\rho'}{3\rho};$$

$$\therefore \qquad \frac{\rho}{\rho'}=\frac{PK}{3\ KM}.$$

(2) Referred to the tangent and normal at the point as axes, the equation of the curve may be written in the form,

$$y=ax^2+2hxy+by^2+\text{higher powers of } x \text{ and } y. \qquad \dots \quad (1)$$

The equation of the chord is $y=k$, say.

The points in which $y=k$ cuts the curve are given by

$$k=ax^2+2hkx+bk^2,$$

if we neglect higher powers of x and y than the second.

$$\therefore \qquad x_1+x_2=-\frac{2hk}{a}.$$

Hence, the co-ordinates of the mid-point of the chord, are $(-hk/a, k)$

Now, by Q. 470, the ratio of the radius of curvature at P to that at the corresponding point of the evolute is given by

$$\frac{\rho'}{\rho}=\frac{3y_1 y_2^2-y_3(1+y_1^2)}{y_2^2}.$$

To find y_1, y_2, y_3 at the origin, differentiate (1) successively;

$$y_1=2ax+2hy+2hxy_1+2byy_1$$
$$y_2=2a+2hy_1+2hxy_2+2hy_1+2byy^2+2by_1^2$$
$$y_3=2hy_2+2hxy_3+2hy_2+2hy_2+2byy_3+2by_1y_2+4by_1y_2$$

Thus $(y_1)_0=0$, $(y_2)_0=2a$, $(y_3)_0=12\,ah$,

and the ratio $$\frac{\rho'}{\rho}=\frac{-12ah}{4a^2}=-\frac{3h}{a}.$$

But $$\mathrm{KM}=-\frac{hk}{a},\ \text{and PK}=k.$$

Hence $$\frac{\mathrm{PK}}{3\ \mathrm{KM}}=-\frac{a}{3h}=\frac{\rho}{\rho'}.$$

Question 500.

(V. K. Aravamudan) :—Solve the equation

$$\frac{dy}{dx}+y^2+\frac{2y}{x}+x^2-\frac{2}{x^2}=0.$$ [illegible] B.D.L. C.B.F.R. P.D.D

Solution by K. J. Sanjana, M.A., and T. P. Trivedi, M.A., LL.B.

Putting $y=\frac{1}{u}\frac{du}{dx}$ and simplifying, we get

$$x^2\frac{d^2u}{dx^2}+2x\frac{du}{dx}+(x^4-2)u=0.$$

To solve this we take $u=a_1x^m+a_2x^{m+r}+a_3x^{m+s}+\ldots$, where m is the lowest exponent of x and r, s, ...are positive quantities in ascending order. The substitution gives

$$a_1x^m\{m(m-1)+2m-2\}+a_2x^{m+r}\{(m+r)(m+r-1)+2(m+r)-2\}$$
$$+a_3x^{m+s}\{(m+s)(m+s-1)+2(m+s)-2\}+\ldots\ldots$$
$$+a_1x^{m+4}+a_2x^{m+4+r}+a_3x^{m+4+s}+\ldots=0.$$

As $a_1\neq0$, we get $m^2+m-2=0$, giving $m=1$ or -2. A little consideration shows that r must be equal to 4; for, if it were 1, 2, or 3, a_2 would be zero and the term in x^{m+r} would be wanting; so also s, t,...are 8, 12,......Thus we have

$$a_2\{(m+r)(m+r-1)+2(m+r)-2\}+a_1=0;$$

and more generally

$$a_{k+1}\{(m+kr)(m+kr-1)+2(m+kr)-2\}+a_k=0,$$

whence $a_{k+1}=-a_k\div\{(m+kr+2)(m+kr-1)\}$.

When $m=1$, $a_2=-\frac{a_1}{4\cdot 7}$, $a_3=-\frac{a_2}{8\cdot 11}=\frac{a_1}{4\cdot 7\cdot 8\cdot 11}, a_4=-\frac{a_1}{4\cdot 7\cdot 8\cdot 11\cdot 12\cdot 15}$, and so on. This gives $u=a_o\left\{x-\frac{x^5}{4\cdot 7}+\frac{x^9}{4\cdot 7\cdot 8\cdot 11}-\dots\right\}=a_0u_1$, suppose.

Similarly, when $m=-2$ we get $b_0u_2=b_0\left\{x^{-2}-\frac{x^2}{1\cdot 4}+\frac{x^6}{1\cdot 4\cdot 5\cdot 8}-\dots\right\}$: so that x is seen to be $a_0u_1+b_0u_2$.

Hence $y=\frac{du}{dx}\div u=\frac{a_0u'_1+b^0u'_2}{a_ou_1+b_0u_2}$, where the dashes denote differentials with regard to x. This may finally be written

$$y=\frac{cu_1'+u_2'}{cu_1+u_2},$$

where $c\equiv\frac{a_0}{b_0}$ is the single constant required by the proposed equation of the first order.

Question 501.

(K. J. SANJANA, M.A.) :—If $(r, n)=1^r+2^r\dots+n^r$, shew that $(r,n)-{}_{r+1}C_1(r, n-1)+{}_{r+1}C_2(r, n-2)-\dots(-1)^{r+1}{}_{r+1}C_{r+1}(r, n-r-1)=r\,!$ where n is any integer greater than $r+1$. When $n=r+1$, or r, prove that the formula is still true, if the last or the last two terms are neglected; and find the value of the series when $n=r-1$.

Solution by R. Vythynathaswamy.

$$r\,!=\Delta^r(n-r)=n^r-{}_rC_1(n-1)^r+{}_rC_2(n-2)^r\dots+{}_rC_r(n-r)^r \quad \dots \quad (1)$$

Also, the left-hand side of the question proposed may be written as

$$\begin{aligned}&n^r-{}_rC_1(n-1)^r+{}_rC_2(n-2)^r\dots\dots\dots\pm{}_rC_r(n-r)^r.\\&\quad -{}_rC_o(n-1)^r+{}_rC_1(n-2)^r\dots\dots\dots\dots\pm{}_rC_r(n-r-1)^r\\&+(n-1)^r-{}_rC_1(n-2)^r+{}_rC_2(n-3^r\dots\dots\\&\quad -{}_rC_o(n-2)^r+{}_rC_1(n-3)^r\dots\dots\dots\\&\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\\&\quad +2^r-{}_rC_1.1^r.\\&\qquad -{}_rC_o\cdot 1^r\\&\quad +1^r.\end{aligned}$$

Hence the left-hand side$=n^r-{}_rC_1(n-1)^r+{}_rC_2(n-2)^r\dots+r\,!$ by (1).

In order that all the terms in the given series may be significant, $n>r+1$,

If $n=r+1$, or, $<r+1$, some of the terms at the end become zero, and some impossible.

When $n=r-1$, the last three terms are not significant.

Question 504.

(R. Srinivasan, M.A.):—Shew that

$$\int_0^\pi \theta \sin\theta \cos^{2m}\theta \cos(a\cos\theta)d\theta = \frac{\pi}{a^{2m+1}}\left\{P\sin a + Q\cos a\right\},$$

where $P=a^{2m}-2m(2m-1)a^{2m-2}+2m(2m-1)(2m-2)(2m-3)a^{2m-4}-\ldots$

$Q=2ma^{2m-1}-2m(2m-1)(2m-2)a^{2m-3}+\ldots\ldots\ldots$

Solution (1) *by T. P. Trivedi, M.A., L.L.B. and K. J. Sanjana, M.A.* (2) *by R. Vythyanathaswamy.*

(1) Let $I=\int_0^\pi \theta\sin\theta\cos^{2m}\theta\cos(a\cos\theta)d\theta$

$$=\int_0^\pi (\pi-\theta)\sin\theta\cos^{2m}\theta\cos(a\cos\theta)d\theta.$$

$$\therefore\quad 2I=\pi\int_0^\pi \sin\theta\cos^{2m}\theta\cos(a\cos\theta)\,d\theta$$

$$=\pi\left\{\left[-\frac{\sin(a\cos\theta)}{a}\cos^{2m}\theta\right]_0^\pi -\frac{1}{a}\int_0^\pi 2\,m\cos^{2m-1}\theta\sin\theta\sin(a\cos\theta)\,d\theta\right\}$$

$$=\pi\left\{\frac{2\sin a}{a}-\frac{2m}{a}\int_0^\pi \cos^{2m-1}\theta\sin\theta\sin(a\cos\theta)d\theta\right\}$$

$$=\pi\left\{\frac{2\sin a}{a}-\left[\frac{2m}{a}\frac{\cos^{2m-1}\theta\cos(a\cos\theta)}{a}\right]_0^\pi -\frac{2m(2m-1)}{a^2}\int_0^\pi\cos^{2m-2}\theta\sin\theta\cos(a\cos\theta)\,d\theta\right\}$$

$$=\pi\left\{\frac{2\sin a}{a}+\frac{2\times 2m\cos a}{a^2} -\frac{2m(2m-1)}{a^2}\times\int_0^\pi\cos^{2m-2}\theta\sin\theta\cos(a\cos\theta)d\theta\right\}$$

$$=2\pi\left\{\frac{\sin a}{a}+\frac{2m\cos a}{a^2}-\frac{2m(2m-1)}{a^3}\sin a -\frac{2m(2m-1)(2m-2)}{a^4}\cos a+\ldots\right\}$$

$\therefore \qquad I=\frac{\pi}{a^{2m+1}}\left\{P \sin a+Q \cos a\right\},$

where P and Q have the values stated in the question.

(2) We have $\int_{-1}^{+1} \sin mx \, dx=0.$

Differentiating with respect to m successively

$$\int_{-1}^{+1} x^{2r+1} \cos mx dx+0.$$

Thus $\qquad \int_{-1}^{+1} \cos mx \sin^{-1}(nx)dx=0,$

Again

$$\int_{-1}^{+1} \cos mx \cos^{-1}(nx)dx=\pi\frac{\sin m}{m},$$

putting $\cos^{-1}nx, =\frac{\pi}{2}-\sin^{-1}nx$, and expanding the inverse function in terms of odd powers of x.

Now, consider the integral

$$I=\int_0^{\pi}\theta \sin \theta \cos (\alpha \cos \theta)\, d\theta.$$

This is the same as $\int_{-1}^{+1} \cos^{-1}x \cos(\alpha x)dx$, where $x=\cos\theta$.

$\therefore \qquad I=\frac{\pi \sin \alpha}{\alpha}$ by (i).

Differentiate this result $2m$ times with respect to α, and we get

$$(-)^m\int_0^{\pi}\theta \sin \theta \cos (\alpha \cos \theta) \cos^{2m}\theta\, d\theta=\pi\frac{d^{2m}}{d\alpha^{2m}}\left(\frac{\sin \alpha}{\alpha}\right).$$

By Leibnitz' Theorem the right hand side is

$$\pi\left\{(-)^m \sin \alpha\cdot\frac{1}{\alpha}+2m\, C_1\cdot(-)^m\cos\alpha\cdot\frac{\underline{|1}}{\alpha^2}\right.$$
$$\left.+2m\, C_2(-)^{m-1} \sin \alpha\cdot\frac{\underline{|2}}{\alpha^3}\ldots \qquad \ldots +\frac{\underline{|2m}}{\alpha^{2m+1}}\right\}.$$

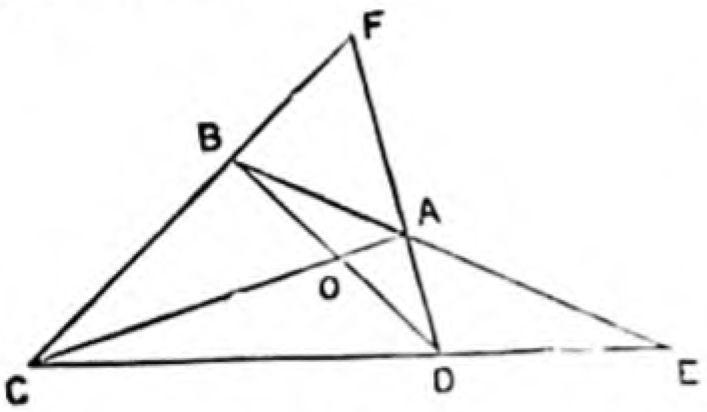
Hence, removing $(-)^m$, we have

$$I=\frac{\pi}{\alpha^{2m+1}}(P \sin \alpha+Q \cos \alpha),$$

where P and Q have the values given.

Question 506.

(A. C. L. Wilkinson, M.A., F.R.A.S.):—Show that it is impossible to describe a plane quadrilateral such that the angle between the diagonals is equal to each of the angles between the two pairs of opposite sides.



Solution by K. J. Sanjana, M. A. and several others.

The angle AOD may be equal to the angle F; but it is$>$the angle E, as it is$>$BDC, which itself is$>$E. Similarly, AOB may be equal to E, but must be$>$F. Hence the result stated.

Question 507.

(S. Ramanujan):—Solve completely $x^2=y+a$, $y^2=z+a$, $z^2=x+a$, and hence shew that

$$(a)\quad \sqrt{8-\sqrt{8+\sqrt{8-\text{etc}}}} = 1+2\sqrt{3}\sin 20^\circ$$

$$(b)\quad \sqrt{11-2\sqrt{11+2\sqrt{11-\text{etc}}}} = 1+4\sin 10^\circ$$

$$(c)\quad \sqrt{23-\sqrt{23+2\sqrt{23+2\sqrt{23-2\sqrt{23+\text{etc}}}}}}$$
$$=1+4\sqrt{3}\sin 20^\circ.$$

Solution by the Proposer.

We have,

$$x=z^2-a=(y^2-a)^2-a=\{(x^2-a)^2-a\}^2-a.$$

or $$x^8-4x^6a+2x^4(3a^2-a)-4x^2(a^3-a^2)-x+a^4-2a^3+a^2-a=0 \ldots \quad (1)$$

Evidently y and z also are the roots of this equation.

But x may be written as

$$\sqrt{a+y}=\sqrt{a+\sqrt{a+z}}$$
$$=\sqrt{a+\sqrt{a+\sqrt{a+x}}}=$$
$$\sqrt{a+\sqrt{a+\sqrt{a+\&c}}}=\sqrt{a+x}.$$

Hence we have $$x^2-x-a=0 \qquad \dots \qquad \dots \qquad (2)$$

Therefore x^2-x-a must be a factor of the expression in (1).

Now, dividing (1) by x^2-x-a, we have

$$x^6+x^5+x^4(1-3a)+x^3(1-2a)+x^2(1-3a+3a^2)$$
$$+x(1-2a+a^2)+(1-a+2a^2-a^3)=0. \qquad \dots \qquad (3)$$

Let $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$, be the roots, of this equation.

Then since y and z are also roots we may suppose

that $$\left.\begin{array}{l}\alpha^2=a+\beta\\ \beta^2=a+\gamma\\ \gamma^2=a+\alpha\end{array}\right\} \qquad \left.\begin{array}{l}\alpha'^2=a+\beta'\\ \beta'^2=a+\gamma'\\ \gamma'^2=a+\alpha'\end{array}\right\}$$

Let $\alpha+\beta+\gamma=u$ and $\alpha'+\beta'+\gamma'=v$, then we see that

$$\alpha^2+\beta^2+\gamma^2=3a+u,$$

that is $$\alpha\beta+\beta\gamma+\gamma\alpha=\frac{u^2-u-3a}{2}.$$

Again we have,

$$\left.\begin{array}{r}\alpha^2\beta=a\beta+\beta^2\\ \beta^2\gamma=a\gamma+\gamma^2\\ \text{and } \gamma^2\alpha=a\alpha+\alpha^2\end{array}\right\} \quad \text{and} \quad \left.\begin{array}{r}\gamma\alpha^2=a\gamma+\beta\gamma\\ \alpha\beta^2=a\alpha+\gamma\alpha\\ \beta\gamma^2=a\beta+\alpha\beta\end{array}\right\}.$$

Adding up all the six results we have

$$\Sigma\alpha^2(\beta+\gamma)=2a(\alpha+\beta+\gamma)+$$
$$\alpha^2+\beta^2+\gamma^2+\alpha\beta+\beta\gamma+\gamma\alpha\,;$$

i.e., $$(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)-3\alpha\beta\gamma=$$
$$2a(\alpha+\beta+\gamma)+(\alpha^2+\beta^2+\gamma^2)+(\alpha\beta+\beta\gamma+\alpha\gamma).$$

i.e., $$u.\ \tfrac{1}{2}\,(u^2-u-3a)-3\alpha\beta\gamma=2au+(3a+u)+\tfrac{1}{2}\,(u^2-u-3)$$

Hence we have $\alpha\beta\gamma=u^3-2u^2-7au-u-3a.$

Similarly we have

$$\alpha'+\beta'+\gamma'=v$$
$$\alpha'\beta'+\beta'\gamma'+\gamma'\alpha'=\tfrac{1}{2}\,(v^2-v-3a)$$
$$6\ \alpha'\beta'\gamma'=v^3-2v^2-7av-v-3a.$$

Hence the sextic in (3) is identical with

$$\{\,x^3-x^2u+\tfrac{1}{2}x(u^2-u-3a)-\tfrac{1}{6}\,(u^3-2u^2-7au-u-3a)\,\}$$
$$\times\,\{\,x^3-x^2v+\tfrac{1}{2}\,x\,(v^2-v-3a)-\tfrac{1}{6}(v^3-2v^2-7av-v-3a)\,\} \qquad \dots \qquad (4)$$

Now equating the co efficients of x^5 as well as x^3 in (3) and (4) we have

$$u+v=-1$$
$$u^3+v^3-2(u^2+v^2)-(7a+1)(u+v)-6a+$$
$$3\,\{\,uv(u+v)-2uv-3a(u+v)\,\}=6(2a-1).$$

Substituting for $u+v$ in the above result, we have, $uv=2-a$.

Hence
$$\left.\begin{aligned} u&=-\frac{1+\sqrt{4a-7}}{2} \\ v&=-\frac{1-\sqrt{4a-7}}{2} \end{aligned}\right\} \quad \ldots \quad \ldots (5)$$
and

These values of u and v substituted in (4) reduce the sextic equation in (3) to the two cubics:

$$x^3+x^2\frac{1-\sqrt{4a-7}}{2}-x\frac{2a+1+\sqrt{4a-7}}{2}+\frac{a-2+a\sqrt{4a-7}}{2}=0,$$

$$x^3+x^2\frac{1+\sqrt{4a-7}}{2}-x\frac{2a+1-\sqrt{4a-7}}{2}+\frac{a-2-a\sqrt{4a-7}}{2}=0,$$

which can be solved by the usual methods.

In the numerical examples proposed, the combinations of the signs *plus* and *minus* may be determined by proceeding as follows :—

$$1+2\sqrt{3}\sin 20^\circ=\sqrt{1+4\sqrt{3}\sin 20^\circ+12\sin^2 20^\circ}$$

$$=\sqrt{7+4\sqrt{3}\sin 20^\circ-6\cos 40^\circ}$$

$$=\sqrt{7+4\sqrt{3}\sin 20^\circ-4\sqrt{3}\cos 30^\circ\cos 50^\circ}$$

$$=\sqrt{7+4\sqrt{3}\sin 20^\circ-2\sqrt{3}\cos 70^\circ-2\sqrt{3}\cos 10^\circ}$$

$$=\sqrt{7+2\sqrt{3}\cos 70^\circ-2\sqrt{3}\cos 10^\circ}$$

$$=\sqrt{7-4\sqrt{3}\sin 30^\circ\sin 40^\circ}=\sqrt{8-(1+2\sqrt{3}\sin 40^\circ)}$$

$$=\sqrt{8-\sqrt{8+(2\sqrt{3}\sin 80^\circ-1)}}$$

$$=\sqrt{8-\sqrt{8+\sqrt{8-(1+2\sqrt{3}\sin 20^\circ)}}}.$$

In a similaar manner we have,

$$1+4\sin 10^\circ=\sqrt{11-2(1+4\sin 50^\circ)}$$

$$=\sqrt{11-2\sqrt{11+2(4\sin 70^\circ-1)}}$$

$$=\sqrt{11-2\sqrt{11+2\sqrt{11-2(1+4\sin 10^\circ)}}},$$

and also

$$1+4\sqrt{3}\sin 20^\circ=\sqrt{23-2(4\sqrt{3}\sin 80^\circ-1)}$$

$$=\sqrt{23-2\sqrt{23+2(1+4\sqrt{3}\sin 40^\circ)}}$$

$$=\sqrt{23-2\sqrt{23+2\sqrt{23+2(1+4\sqrt{3}\sin 20^\circ)}}}.$$

QUESTIONS FOR SOLUTION.

532. (R. Vythynathaswamy) :—If 2A, 2B, 2C are the mutual inclinations of any three lines in space and 2L, 2M, 2N, the angles which any fourth straight line makes with them, shew that the following identical relation holds

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 4 \\ 1 & 0 & \sin^2 C & \sin^2 B & \sin^2 L & 1 \\ 1 & \sin^2 C & 0 & \sin^2 A & \sin^2 M & 1 \\ 1 & \sin^2 B & \sin^2 A & 0 & \sin^2 N & 1 \\ 1 & \sin^2 L & \sin^2 M & \sin^2 N & 0 & 1 \\ 4 & 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0.$$

533. (Prof. K. J. Sanjana.)—Three real numbers $a, b, c,$ are written down at random. Find the chance (i) that $b^2 < 4ac$, (ii) that $b^2 > 4ac$. What is the chance of b^2 being equal to $4ac$?

534. (Zero) :—A given ellipse is projected orthogonally on a given plane. Shew how to obtain the axes of the projection geometrically.

535. (N. P. Pandya) :—Construct a conic having given three points on it and the symmedian point and orthocentre of the triangle formed by the tangents at those points.

536. (K. Appukuttan Erady, M. A.) :—Prove that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ue^{-u'}\,dx\,dy = \frac{\pi}{2}\,\frac{(ab'+a'b-2hh')}{(a'b'-h'^2)^{\frac{3}{2}}},$$

where $u \equiv ax^2+2hxy+by^2$ and $u' \equiv a'x^2+2h'xy+by^2$.

537. (K. J. Sanjana, M.A.) :—Prove that

$$\frac{1}{2}\,\frac{x^2}{2} - \frac{1\cdot 3}{2\cdot 4}\,\frac{x^4}{4} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\,\frac{x^6}{6} - \ldots\ldots$$

$$= \tfrac{1}{4}\log(1+x^2) + \tfrac{1}{8}\log(1+x_1^2) + \tfrac{1}{16}\log(1+x_2^2) + \ldots,$$

where $x_1^2 = \dfrac{x^4}{4(x^2+1)},\ x_2^2 = \dfrac{x_1^4}{4(x_1^2+1)},\ \ldots\ldots$;

and that $\quad \frac{1}{2}\frac{x^2}{2}+\frac{1\cdot3}{2\cdot4}\frac{x^4}{4}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\frac{x^6}{6}+\quad\ldots$

$$=-\tfrac{1}{4}\log(1-x^2)-\tfrac{1}{8}\log(1+x_1^2)-\tfrac{1}{16}\log(1+x_2^2)-\ldots$$

where $\quad x_1^2=\frac{x^4}{4(1-x^2)},\ x_2^2=\frac{x_1^4}{4(1+x_1^2)},\ \ldots\ldots\ldots$

In the first equality x is any real quantity; in the second, any real quantity <1.

538. (A. NARASINGA RAO):—If the tangents at the circular points of a nodal circular cubic curve meet at P on the curve, then any transversal through P cuts the curve again in pairs of points such that the lines joining them to the node form a pencil in orthogonal involution.

539. (S. KRISHNASWAMI AIYANGAR):—The focal chords of the maximum inscribed ellipse of the triangle of reference parallel to the sides are the roots of the equation

$$x^3-2x^2\Delta^{\frac{1}{2}}\tan^{\frac{1}{2}}\lambda\cot w+x\Delta\tan\lambda\operatorname{cosec}^2 w-2R^2\Delta^{\frac{1}{2}}\tan^{\frac{3}{2}}\lambda=0,$$

where λ is the minor semi-steiner angle, w is the Brocard angle and R the circumradius.

540. (S. KRISHNASWAMI AIYANGAR):—If S, H be the foci of the maximum inscribed ellipse of ABC and SH subtends angles $\theta_1, \theta_2, \theta_3$ at the vertices, prove that

$$bc\cot\theta_1+ca\cot\theta_2+ab\cot\theta_3=\Sigma(a^2)-\{\Sigma(a^4)-\Sigma(a^2b^2)\}^{\frac{1}{2}}.$$

541. (S. RAMANUJAN):—Prove that

$$1+\frac{1}{1\cdot3}+\frac{1}{1\cdot3\cdot5}+\frac{1}{1\cdot3\cdot5\cdot7}+\ldots+\frac{1}{1+}\,\frac{1}{1+}\,\frac{2}{1+}\,\frac{3}{1+}\,\frac{4}{1+}\ldots=\sqrt{\left(\frac{\pi e}{2}\right)}$$

542. (K. V. ANANTANARAYANA SASTRI, B.A.):—Expand $\theta\cot\frac{1}{4}\theta$ in powers of $\cos\theta$.

543. (R. VYTHYNATHASWAMY):—Shew that

$$1^r+2^r+3^r+\ldots\ldots n^r=(c_2\Delta+c_3\Delta^2+\ldots\ldots c_{r+1}\Delta^r)0^r,$$

where $c_p={}_{n+1}C_p$.

544. (SELECTED):—Evaluate the definite integrals

$$\int_0^\infty e^{-x^2}\sin ax^2dx;\ \int_0^\infty e^{-x^2}\cos ax^2dx,$$

545. (M. T. Naraniengar, M.A.) :—Shew that the roots of $8x^3-18x+9=0$, and $8y^3-18y-9=0$, are connected by the relations

$$x=3+2y-2y^2\text{; and } y=3-x-2x^2.$$

546. (S. Ramanujan) :—Shew that

$$\text{(i)}\quad \left(\frac{1}{3}-\frac{1}{4}\right)+\frac{2}{3^2}\left(\frac{1}{3}-\frac{1}{4^2}\right)+\frac{2\cdot4}{3\cdot5^2}\left(\frac{1}{3}-\frac{1}{4^3}\right)+\ldots\ldots\ldots$$

$$=\frac{\pi}{12}\log\left(2+\sqrt{3}\right)$$

$$\text{(ii)}\quad 1-\frac{2}{3^2}+\frac{2\cdot4}{3\cdot5^2}-\frac{2\cdot4\cdot6}{3\cdot5\cdot7^2}+\ldots\ldots\ldots=\frac{\pi^2}{8}-\tfrac{1}{2}\left\{\log\left(1+\sqrt{2}\right)\right\}^2$$

547. (V. K. Aravamudan :—Prove that the oblique trajectories of the system of curves represented by the elimination of t between

$$x=f(t,a),\ y=\phi(t,a)$$

a being the parameter of the family, are given by replacing a by the complete integral of

$$n\left\{\left(\frac{\partial f}{\partial t}\right)^2+\left(\frac{\partial \phi}{\partial t}\right)^2\right\}+\frac{\partial a}{\partial t}\left\{n\left[\frac{\partial \phi}{\partial t}\frac{\partial \phi}{\partial a}+\frac{\partial f}{\partial t}\frac{\partial f}{\partial a}\right]\right.$$

$$\left.-\frac{\partial \phi}{\partial t}\frac{\partial f}{\partial a}+\frac{\partial f}{\partial t}\frac{\partial \phi}{\partial a}\right\}=0$$

and hence find the trajectories of a system of confocal ellipses, n being equal to the tangent of the angle of intersection.

List of Periodicals Received.

(From 16th January to 15th March 1914.)

1. American Journal of Mathematics, Vol. 36, No. 1, January 1914.
2. Astrophysical Journal, Vols. 38 and 39, Nos. 5 and 1, December 1913 and January 1914.
3. Bulletin of the American Mathematical Society, Vol. 20, Nos. 4 and 5, January and February 1914.
4. Bulletin des Sciences Mathematiques, Vol. 35, January & February 1914.
5. Crelle's Journal, Vol. 144, No. 1, January 1914.
6. Educational Times, February 1914, (6 copies)
7. L'Education, Mathematique, Vol. 16, Nos. 7, 8 and 9.
8. Fortschritte der Mathematik, Vol. 42, No. 2.
9. L'Intermediaire des Mathematiciens, Vol. 20, Nos. 11 and 12, November and December 1913.
10. Journal de Mathematiques, Elementaires, Vol. 38, Nos. 7, 8 and 9.
11. Liouville's Journal, Vol 9, No. 4.
12. Mathematical Gazette, Vol. 7, No. 109, January 1914, (3 copies).
13. Mathematical Reprints from Educational Times, Vol. 24.
14. Mathematics Teacher, Vol. 6, No. 2, December 1913.
15. Mathematische, Annalen, Vol. 75, No. 1, February 1914.
16. Mathesis, Vols. 3 and 4, January to December 1913 and January 1914.
17. Messenger of Mathematics, Vol. 43, Nos. 7 and 8, November and December 1913.
18. Monthly Notices of the Royal Astronomical Society, Vol. 74, Nos. 1 & 2, November and December 1913.
19. Philosophical Magazine, Vol. 27, Nos. 157 and 158, January and February 1914.
20. Popular Astronomy, Vol. 22, Nos. 1 and 2, January & February 1914.
21. Proceedings of the London Mathematical Society, Vol. 13, No. 1, December 1913.
22. Proceedings of the Royal Society of London, Vol. 89, Nos. 612 and 613, January and February 1914.
23. Quarterly Journal of Mathematics, Vol. 45, No. 1, December 1913.
24. Revue de Mathematiques Speciales, Vol. 24, Nos. 4 and 5, January and February 1914.
25. School Science and Mathematics, Vol. 14, Nos. 1 and 2, January and February 1914, (3 copies).
26. Transactions of the Royal Society of London, Vols. 213 & 214, Nos. 505, 506, 507, 508 and 509.
27. The Tohoku Mathemical Journal, Vol. 4, Nos. 1, 2 and 3, September and December 1913.
28. Nature, Vol. 92, Nos. 2305, 2306, 2307, 2308 and 2309, January 1914.
29. Rendiconti del Circolo Mathematico Di Palermo, Vols. 36 and 37, Nos. 1, 2 and 3 and 1 and 2.

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CONTENTS.

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Contributors will be supplied, if so desired, with extra copies of their contributions at net cost.

All contributions should be written legibly on one side only of the paper, and all diagrams should be given in separate slips.

All communications intended for the Journal should be addressed to the Hony. Joint Secretary, M. T. NARANIENGAR, M.A., Mallesvaram, Bangalore.

All business communications regarding the Journal should be addressed to the Hony. Asst. Secretary, P. V. SESHU AIYAR, B.A., L.T., 37, Venkatachala Chetty Street, Triplicane, Madras.

Enquiries by intending members and all other communications may be addressed to the Hony. Joint Secretary, D. D. KAPADIA, M.A., B.Sc., 322, Main Street, Poona.

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PROGRESS REPORT.

Mr. M. K. Kewalramani, B.A.—M.A. student, Fergusson College, Poona, has been elected a member of our Society at concessional rate.

2. Our Treasurer, Rev. C. Pollard, is proceeding to England on leave, and the Committee therefore thought it advisable to appoint Mr. S. Narayana Aiyar, M.A., F. S. S., Manager, Port Trust Office, Madras, to take up the duties of the Treasurer. The attention of the General Body is drawn to this change so that all matters pertaining to the Treasurer, may be referred to Mr. S. Narayana Aiyar during Mr. Pollard's absence.

3. *A Junior Trigonometry*—by Messrs. Borchardt and Perrott, has been received from Messrs. G. Bell & Sons, London.

The University of Illinois has presented to our Library the reprints of the Ph. D. Theses on the following subjects—

1. Invariants of Linear Differential Equations.
2. Projective Differential Geometry of Developable Surfaces.
3. Primitive Groups, with a determination of the Primitive Groups of Degree 20.
4. Functions of Three Real Variables.

Reprints of papers by Prof. G. A. Miller on Substitution Groups not exceeding the 7th degree, Groups generated by the two operators satisfying the condition $s_1s_2=s^{-1}s''_1$, Gauss's Lemma and some related Group Theory, Groups which contain an Abelian subgroup of prime index, on the use of Co-sets of a Group, Infinite Systems of Indivisible Groups, a Third Generalization of the Groups of the Regular Polyhedrons and some minor pamphlets have likewise been received.

POONA, 31st May 1914. D. D. KAPADIA, *Hony. Joint Secretary.*

Mathematics in Relation to other Subjects of Knowledge.*

By Prof. R. Littlehailes, M.A.

On accepting the invitation to deliver your inaugural address my first feeling was one of cheery optimism, the subjects at my disposal being so many and varied that I felt certain of easily choosing one that would be acceptable to you all. On second thoughts however, numerous difficulties presented themselves. The subject of this inaugural address should be one that is fairly general in character, it should not be too technical, it should not be beyond the comprehension of even the least mathematically inclined of the audience and yet it should bear on the subject—mathematics—which offers the excuse for the existence of this Society.

I cogitated whether it would not be well to select the life of some renowned mathematician like Newton, Descartes, Euler, Lagrange or Laplace and give a sketch of his life and of the influence that he has exercised in the subject that we have so much interest in. Second thoughts, however, led me to choose the present title for the subject of my address this evening; in the first place because it is a relation that has always had a keen interest for me, in the second place because it is a subject that has reference to the *rationale* of the foundation of this Society, and thirdly because it is a subject that offers scope for exciting the attention not only of those who like ourselves take the subject of mathematics seriously, but also of those who by virtue of the modern necessities of their own particular special studies require its aid in their own work in a thousand and one ways.

There appears to me several ways of treating the subject of this address, but two of them stand out before the rest. I might narrate *seriatim* the various branches of human knowledge and ignorance and specify in detail the various branches of mathematics that are applicable to their complete study and research. On the other hand, I might catalogue the various branches and sub-divisions of mathematics—both *pure* and what is technically known as *applied*—and consider how each division or sub-division has been utilized by searchers after truth in other branches of knowledge; I have not adopted either of these methods to the exclusion of the other, but have compromised by accepting *in toto* neither the former nor the latter mode of exposition.

I have presumed, of course, that mathematics *has some* relation to other subjects of knowledge, the presumption appearing to me so obvi-

* An address delivered at the inaugural meeting of the Mathematical Association of the Presidency College, Madras, 1914.

ous as hardly to require notice. I might even go so far as to state that I am willing to show the applicability or relationship that exists or might exist between it and any other subject that might at random be named. Having that in mind it is all the more difficult for me to render this address in any way complete or systematically detailed. It must be and must remain—unless you were willing to sit for a course of lectures on this subject—of a cursory character; more of the nature of a brief survey, than of an exhaustive treatise wherein logical sequence of unimpeachable reasoning is the main criterion of excellence.

And this brings me naturally to one of the first divisions of my subject—the reasoning powers of mathematicians and the more general question of the philosophy of reason: a subject that is no doubt often well debated by philosophers who know no mathematics, though the converse, *viz.*, the mathematician who knows no philosophy, scarcely exists; for, who can be a mathematician without being at the same time a philosopher, a lover of wisdom, even if it be but a small portion of the wisdom of the universe? One of the recent tendencies of modern mathematics is the intense study of first principles, a profound introspection and critical examination of the foundation principles of the subject; its fundamental concepts and methods have been subjected to most careful scrutiny and the part played by in tuition and by logic, with the necessary limitation to each, have received the most careful attention of several of our most renowned thinkers, in particular of Whitehead and Russell. Their written cogitations take us immediately into the realms of "Metaphysics"—the careful preserve of the philosopher. This subject has been variously defined, by both the more and the less seriously inclined; its abstruseness is probably well represented by the cynical definition that the study of metaphysics is akin to "a blind man searching in a dark room for a black cat which is not there." And to quote merely one of its applications to mathematics, I might state that we involve ourselves in apparently contradictory but in reality absolutely comprehensive statements: any connection between a and b may be considered as a relationship between a and b; if there be no connection between a and b, the statement that there is no connection is itself the statement of a relationship between them. This of course means that there is nothing in nature which does not bear some relationship to any or every other thing in nature,—a negative attribute being as essential, as a positive attribute, in the definition and limitation of a relationship. I do not however desire to dwell upon the relation of mathematics to philosophy and in particular upon that portion of the philosophy of mathematics which makes it self-evident

that mathematics is related to every other subject upon which concepts may be formed and thoughts uttered.

I shall turn my attention to what I consider, in these essentially practical, industrial and commercial modern times, one of the most important features of mathematics ; *viz.*, its relation to what is currently known as Science. Specific branches of science *e.g.*: the treatment of Dynamics in its extremely large number of branches are recognised subjects of study and research of students of mathematics and to detail every connection between pure mathematics and mechanics would be as tedious as the compilation of a Tamil Dictionary, though possibly much more interesting. I shall however have to touch upon one or two main connections shortly, but before doing so I desire to emphasize the attitude taken up by the more ardent investigators in the various fields of science ; it is that no real and true science can exist until the phenomena pertaining thereto can be and are measured quantitatively and not merely observed qualitatively. That a phenomenon exists and is observed is the first step towards the formulation of any science, but that is far from being the whole method of science. Observations should then be made of the general tendency of the phenomena, whether there are any changes in time, any changes in space, any changes in mass, and when these changes are observed the variations are classified and we reach such a science, as for instance, "systematic botany." There is however no true science until the aid of mathematics is sought and applied. That varieties, variations and changes exist is interesting and these divergencies offer food for a certain amount of more or less inexact thinking and reflection such as a child in its earlier stages of development may be expected to indulge in, but the mature rational mind, especially if that mind has received a logically scientific training, cannot rest content there. It must *measure* the *directions* of these variations and their *magnitudes* both with regard to space and with respect to time ; and it is only at this stage that the phenomena come to be classed as truly scientific and can be treated in a really scientific manner. No real advance can be made into the science of any phenomena until the aid of mathematics is sought.

Some sciences like the so-called natural sciences have come under the influence of mathematics to an extremely small degree, while others like physics are so interlocked with mathematics that the student of physics cannot proceed without calling in the aid of the mathematician at almost every step. Let us glance, for a moment, at one of the natural sciences, geology for example, and let us take one of its sub-

groups, mineralogy. In the treatment of this subject how is mathematics involved? To begin with there are the various types of crystal symmetry which are best discussed about three axes, and the student of analytical geometry of three dimensions looks upon symmetry in crystallography as a mere corollory to some of his more elementary mathematical ideas connected with three dimensional space. Further the determination of different kinds of minerals and in particular of crystals, is simplified by their behaviour with regard to light, and especially as regards their behaviour in polarised light by means of which some of the most beautifully coloured pictures of symmetry and of geometric shapes are evidenced. But how are we to further investigate the formation of these shapes and colours? They are investigated by a consideration of the geometry of curves and surfaces coupled with some theory of light. And how can we investigate or even postulate any theory of light without mathematical analysis at our elbow? Is there any student of physics among you, who can tell me that he has discussed the wave theory of light, refraction, polarisation, wave lengths, interference or any of the current ideas and phenomena of optical science without calling to his aid mathematics? And have not many of you felt that this particular mode of treatment of the subject, or that special little piece of analysis, was difficult, only because it was not completely comprehensible without a prior mathematical training? It is probably too well known among you that the fixation of geological dates is another geological subject that has given rise to mathematical calculations based upon cooling phenomena, and it is needless for me to spend more time on that subject than that taken in mentioning it; Kelvin's writings on the secular cooling of the earth show to geologists and physicists alike the use and application of mathematics to their respective branches of knowledge. I might choose other sub-divisions of geology and explain how they have been shown or might be shown to depend for their comprehensive treatment on a knowledge and application of mathematics; but I must pass on to some other subject.

In physiology we do not rest content until we have measured the *amount* of change in the circulating system due to respiration, the *time* of reflex action and until we have taken other more or less absolute measures. These merely form the basis for other estimates and determinations which have as one of their ultimate objects the meaning of life and death from the physiological standpoint. It would be of extreme interest to show the relationship that exists between mathematics and physiology, but I am not qualified to do more than mention that one of the

modern tendencies therein is, as it is in all other subjects, to demand mathematical exactness in all observations, calculations and inferences.

The science of medicine is more nearly allied to that of chemistry, chemical physiology or physiological chemistry being one of the most important branches of modern science, and so we now turn our attention for a moment to the interesting subject of chemistry which assumes a discussion of the atom as pre-eminently its own. We shall not quarrel with her on that ground. We can well afford to be magnanimous. We know that chemistry without the atom is almost as insipid as rice without curry and so we freely give the atom to the chemist to play with. But he cannot play very long with his atom without experiencing an anxious desire to know something more about it than that it is his fundamental postulate, and he turns to the physicist to aid him in a discussion of its behaviour under certain circumstances and particularly under the action of heat. The physicist is delighted at the honour paid to him (though he considers in his innermost self that it was obvious to even the meanest intellect that his aid was all the while necessary) and sets forth to propound his ideas of the atom and its behaviour. But even he has not gone far before he finds himself in almost as complete a *cul de sac* as the chemist was before him. He finds that he is confronted with certain phenomena which can be very well expressed by means of mathematical equations; but the discussion of these equations, their complete solution and re-interpretation in terms of the original physical concepts are problems that he does not feel quite at home with. He has no other resource than to call in his friend the mathematician who lays the store of his analytical wealth at his feet. Let us make the relationship of chemistry to mathematics a little more concrete by abstracting from any chemical syllabus. I quote from the B.A. syllabus of the University of Madras. "The law of mass-action; the velocity of chemical change; the relations of chemical energy to heat, and to electrical energy,......the kinetic theory of gases." It does not need a specialist to see that in these portions of chemistry the chemist seeks the aid of the physicist and of the mathematician. Even now some on you will have recollections of the haunting fear that is inspired in you at the recollection that thermodynamical formulæ are equations that your lecturer and professor expect you to have heard of, to have seen and to have read about; nay, more, your professor possibly expects you to be acquainted with them; he may even expect you to be so enamoures of them as to spend some odd minutes—hastily snatched from watching an interesting cricket or football match or from participating in a game of tennis—in endeavouring to solve problems on the thermodynamics of

a gas that is not perfect. An awful nightmare it must be to the non-mathematical chemist to have such a vision called up before him.

But the chemist is not alone. The physicist in the advanced stages of his researches grieves at the amount of mathematics that he is required to be acquainted with. He is even required to be at home several afternoons each week to "the calculus" and his friends. Indeed he has come to recognize that he is completely out of the social "world of accurate science" unless he has frequent visits from the universal friend, "mathematical analysis." How could he consider problems on the conduction of heat without "Fourier's Series," and how can Fourier's series be considered if not in connection with general analysis?

I might now turn to one branch of physics that is of so universal an application that we can only just touch upon its fringe. I mean electricity with her sister magnetism. Electricity is such an immense subject that I find some difficulty in knowing what particular branch should be first instanced as leading up to or depending upon mathematics for its investigation. I might instance one of the smallest and yet most important equations that it is the pleasure of the electrician to deal with, I mean the equation of motion of a magnet situated within a coil around which a current momentarily flows and which is represented in various forms but always as a differential equation of the second order. It is indeed the same equation as we meet with in the study of oscillatory motion in rigid dynamics and whose solution denotes either periodic or aperiodic motion. The mathematical study of this equation in its various forms is not difficult and its application to the throw of a galvanometer, to the swing of a pendulum and to many other natural phenomena with which you may or may not be acquainted is extremely interesting, first of all in the application of the equation to the motion under discussion and secondly—and this is perhaps of even more importance—in the interpretation of the solution of the equation in terms of the physical constants from which the original differential equation was formed.

Again, electromagnetic theories open up a large field of mathematical research and are so closely allied to radiation theories—either as offering material for analogy, or as giving opportunity for controversy—that it is no wonder that the advice of the mathematician is sought to reduce the theories to symbols, to criticise the deductions comparing the true with the doubtful, and to state carefully and analytically the suppositions upon which any hypothesis is based and the conclusions to which any accepted hypothesis must lead to. It would be beyond the scope of a general talk of the present nature

to enter at any length upon a discussion of the theories of radiation and electromagnetism; it remains to emphasize the fact that theories based upon experiment and propounded by physicists must be subjected to the critical examination of the mathematical analyst just as the theories that are propounded by the mathematician have to be subjected to the vital tests of conformance with experimental phenomena such as are observed by the physicist.

It is hard to leave the subject of physics in such a discussion as we have before us at present. We could continue with a reference to the theory of sound, its vibrations, its transmission of energy and all its mathematical representations which have been so ably exposed by such masters of the subject as Helmholtz and Rayleigh, both of them mathematicians of no mean order. We could refer to the mathematical theories of the conduction of heat, to the mathematical discussion of elasticity with its accompanying stress and strain about which volumes of mathematics have been written. We could touch upon probably the most interesting of modern mathematical researches—those which bear upon air-ships and promise to revolutinize some of our present means of travel. Aero-dynamics is a fascinating subject from the standpoint of the man in the street; but, the mathematician recognises that the aeronaut desires that his life shall be in no greater danger than that of the motorist and he wishes to trust himself only to such an aeroplane as is proved by mathematical analysis as well as by experiment to be one in which the stability of its motion is assured under even the most adverse conditions. The application of such analysis is only now being made; the theory is, in places, heavy and based upon various suppositions of fluid resistance and strength of materials, but advances are foretold and the aeroplane of the future will without doubt be modelled from data that have mathematics as the guarantee for its stability of motion.

Hydraulics and fluid resistance are met with in the construction of ships and also in a discussion of one of the modern means of developing energy, namely, in the turbine and in particular the steam turbine, while a knowledge of the strength of the materials used in the construction of all machines is necessary; this again reverts to a consideration, among other things, of the elasticity of the substance used in construction. We must not however, at this stage, claim too much for the subject of our heart. Mathematics helps a very great deal in the development of all modern sciences, yet in the commercial and industrial life of the scientific world a great deal of empiricism forms the basis of its constructions. A certain thickness of an iron bar is used

for a certain portion of a machine or building, not because it has been proved mathematically that this thickness is necessary to withstand the strain that is likely to be produced by the stress, but because experience and experiment have proved that this thickness gives a suitable quantity of material able to bear the necessary tension or thrust. Mathematics enters here only by gradually systematising the forces at work upon the several portions of a machine and of reducing these forces with their actions and interactions to their fundamental constituents and indicating the necessary strength of the several parts that is required to withstand the applied forces. And in this there remains an immense amount still to be done by engineering mathematicians and mathematical engineers. One could wish that a larger number of such existed; for it is deplorable to find so much empiricism and acceptance of dubitable authority on the part of engineers due to a great extent to the magnitude of their subject and in part, no doubt, to the disfavour with which many of the most capable mathematicians that the world has produced, look upon the apparent degradation of their higher mental outlook when applied to the ordinary commercial and industrial facts that they so often scorn. It is further to be remembered that a very large number of the applied mathematical problems of to-day did not exist a century ago. With the advent of steam and oil engines and of the application and use of electrical forces, new problems have arisen which are increasing day by day and are of necessity first attacked experimentally as problems of experimental or industrial science before the aid of the mathematician is sought.

It is to be regretted that the mathematical treatment of a subject is in general only sought when it is found that it can make little further advance upon purely empirical and experimental lines. Who, for example, sought to discuss with any fullness the viscosity and lubricating powers of various oils and to what extent variations in surface tension were dependent upon differences of temperature until it became necessary to lubricate fast moving machinery—a product of our more modern industrial civilization? In these discussions mathematical equations whether they are the simple ones of an ordinary readily soluble linear algebraic relation or the somewhat more difficult differential equations arising in oscillatory motion, or in thermodynamics or other relations, and which may not yet be completely solved by the ordinary analysis at our disposal,—are invariably used to represent the physical phenomena observed. Their ultimate solution is necessary and, to my mind, certain.

But let us turn aside for a while to the application of mathematics to subjects of study that do not at first sight appear to allow of any applicability. We may begin with history and economics—two subjects which are usually coupled together though there is no obvious necessity therefor. Mathematics, physics, geology, in fact, any subject of knowledge can be treated historically—just as economics so frequently is. It is to certain aspects of economics that I would invite your attention for a moment. The principles of economics are now being investigated from the standpoint of statistics and the treatment of statistics is essentially a mathematical matter of some importance. In it we are lead to a study of frequency curves and of correlation co-efficients. Modern statistical methods are based upon the theory of probability and are usually associated with the name of Professor Karl Pearson. I have hopes that the University of Madras will see its way, during the next academic year, to invite some statistical authority to Madras for the purpose of delivering a course of lectures of both an elementary and an advanced character on statistical methods. I cannot however do better here than give the ordinary definition of statistics as the science of averages. It would please me to enter into a discussion of averages, modes, means, frequency curves, correlation and probable errors and of their application to the study of economic factors but it will suffice if I refer you to the papers of men such as Pearson, Edgeworth, Elderton, Yule and Bowley. Statistics is a branch of applied mathematics that has unfortunately not received as much attention as it deserves in this country where there are vast accumulations of data awaiting analysis by some one with the necessary mathematical training and with much time at his disposal. Some future research scholars have herein a suggestion for their subject of investigation. "The method of least squares" which used to find so much favour with scientists and indeed at the present day does still find a large amount of adherance on their part is only one method of mathematically treating a series of recorded observations, and it can give place to their treatment by such frequency curves as appear best fitted to the range of observations that may be under discussion.

Questions based on the theory of probability naturally lead us from statistical economics, as distinguished from historical economics, to a matter that we all think of sooner or later, the probability of the length of our life and the precautions that we all should take to ensure that those dependent upon us may at our death not be left unprovided for. The Hindu Joint Family system does not necessitate life insurance and assurance on the part of Hindus to such an extent as it does on the

part of members of other Nationalities, but with the breaking up of the joint family system and the dispersion of the units that form the family, the necessity for such a provision urges itself forward. It is not every one who thinks how it is that one insurance company charges larger premiums than another for the insurance of a life, or how the sum fixed as a premium is arrived at. Considerations both theoretical and experimental govern the selection and treatment of such statistics as form the basis of the various tables that are used in fixing the sums required to be paid as premiums—tables of mortality, of sickness, of succession, of marriage, of superannuation and so forth. All these are to be used by the actuary and must be prepared by him or by his friend the mathematical statistician. Actuarial science may be defined as the collection and utilization of statistics for the purpose of calculating life contingencies—a narrow definition; in a wider sense it is the collection and analysis of past experience and the employment of the results of such analysis to forecast the average future. This analysis may be explained in simple mathematical language as the formation of certain curves which trace out the relationships between the real varibles concerned as deduced for a very large number of observations (*e.g.*, between the age of a person and the number of persons living at that age) and the application of these average curves to any particular case in point. This treatment is essentially mathematical, though the standard of mathematical knowledge required by the student is no exceptionally high and I suggest that some of you might in the near future apply your mathematical knowledge in the direction of statistical analysis.

The hypercritical listener may at this stage ask how mathematics is allied to the study of certain other branches of knowledge, for instance, the study of English and of languages in general. I would not pretend to state other than that the connection is practically nil. We here come to the main distinction between what is known to the educationist as the difference between literary studies and scientific studies. The student of literature may go through the world with his reasoning faculties trained to a very slight extent, the student of science, however cannot make any real satisfactory progress in his work unless his reasoning faculties are developed and one of the surest and safest methods of developing these faculties is by means of an accurate mathematical training. I have long known that the Hindu has many excellent characteristics but that it is necessary for these characteristics to be qualified by a course of mathematical training, so that the average Hindu mind may be reduced from what may be mathematically termed as a tendency to

state of small oscillation about a position of relatively unstable equilibrium to a tendency to a state of steady motion about a position of mental equilibrium. This training may be given by a study of other subjects than mathematics, but the value of these subjects is dependent upon the amount of reasoning that is required from the student, and without doubt this training of the reason is more readily found in a study of mathematics than in any other subject. We frequently hear it stated that the Hindu lacks "common sense." It has been my good fortune to find this remark applicable to extremely few of the Indians with whom I have come into close contact; it may be that the remark is one that has very limited reference, but even if there is the smallest modicum of truth in it, I am convinced that my experience is the result of the fact that "common sense" has been developed by the rational training that every successful student of mathematics is compelled to undergo.

I desire now to touch upon the "non-humanising" influence that a study of mathematics is supposed to have upon all its ardent students. I have heard the same statement made about students of other subjects, *e.g.*, about the student of metaphysical philosophy who enjoys such literature as Kant's *Critique of Pure Reason.* I confess that there appears to be a little, but very little, truth in the idea. Mathematics is essentially the subject of the specialist and every specialist is presumed to be slightly un-human, but I comfort myself with the thought that there is something more than human and the non-humanising influence of mathematics is the development of the human mind beyond that which is human into that which is more than human. On the other hand, if we pick up any book on the "History of Mathematics" and read of the development of mathematics from the earliest recorded times, if we glance into the actual lives of the men who are only known to many of us as mere names, we will find that they lived, ate, drank, married, begot children and died just as we do now-a-days; we will read that they had their worries and trials, their poverty and honour and that they eventually went the way of all flesh, forgotten by the majority of the world and occasionally remembered only by a student as a mere name; as for instance in Newton's Law of motion, in Bessel's function, or in Cartesian co-ordinates. We then experience the feeling that the mathematician cannot be singled out to stand at the corner of the street and thank his God as the Pharisee did, that he is not as other men are.

One fact I should like to impress upon you at this stage; very few of the older students appreciate it and practically none of the younger students know it. It is that mathematics like all other sciences is a

living science. The development of this idea would occupy too much time and I would only just mention one way in which this is brought home to us. The inter-dependence of pure and applied mathematics, as for instance in the application of mathematics to questions arising out of the study of natural phenomena, frequently produces a problem which is mathematically insoluble by the methods of pure mathematics now at our disposal. This necessitates therefore further investigations into and developments of pure mathematics, and in particular of mathematical analysis; thus the progress of mathematical science is emphasised and history is made.

Another branch of science to which I might invite your attention as a subject of investigation is meteorology. Those of you who have been sufficiently interested in this fascinating subject will have recognized that until very recently practically all our observations have been made in two dimensional space. Much can be made and indeed has been made of an analysis of meteorological data of two dimensions but there is apparently no hope of completeness or finality in our deductions until we can measure pressures, temperatures and the like, simultaneously at places *above* the surface of the earth, as well as at places *on* its surface. This, you might retort, is the work of the physicist. I agree: but as soon as the observations are made and collected, it behoves the mathematician to step in and proceed with his analysis of the amassed figures. Meantime, many of the figures collected for stations in India on the surface of the earth await investigation, and I throw out the suggestion for your application.

But I have occupied your time too long already and must bring these rather scattered remarks to a speedy termination. It would ill become me, however, to close an address of this kind and in this country without some reference to astronomy, the study of which dates back almost as far as the origin of history. To enlarge however on the connection between astronomy and mathematics to this audience would be superfluous and I can only ask you to bear astronomy in your mind as another subject in which the mathematician is absolutely necessary.

Finally, I thank you for the honour you have done me in requesting me to deliver your inaugural address and I conclude wishing your Society every good wish for its future prosperity.

Efficiency in Elementary Mathematics.*

To-day the dominant note in education is efficiency. It is fitting therefore that we should consider the factors for training in efficiency in elementary mathematics.

The expressions of opinions, serious comment, great thoughts, inventions, social and intellectual achievements are not spontaneous, nor do they occur in a single mind because they are results of race development. How often in our experience have we been spared from expostulating on some cherished idea, design, method or invention because accident has given priority to some other independent worker. Such first announcements are apt to be vague and undefined and as a consequence men take issue and often give way to passionate disregard and intolerance for the tenacious attitude of an adversary.

I have a pretty firm belief in that greatest of paradoxes, *viz.*, that both sides of an argument are right. Therefore, my friends, I am going to regard your assertions relating to the cultural and utilitarian value of any mathematics material as amphibolous aspects of an ideal future vocation. I am not going to reconcile these aspects, because they need no reconciliation. If you will permit me to state an analogy, it appears to me as if some of us had gone into a long glass tube, while others remained outside. In so doing we have chosen our limitations of vision. When we recognize the fragile barrier that separates our groups we can understand why we look upon one another telescopically or microscopically. We owe to a heritage the fallible belief in a dualism which has been symbolized constantly in the development of our literature, art, religion and science. The sense of opposites expressed as north and south, hot and cold, good and evil suggest culture and utility. Let us give praise to that noble soul who first enunciated the law of *relativity.* Now you know there is no greater avenue of adventure than higher geometry which reveals the infinite attenuation of life and comprehends the processes which guide the unborne possibilities of the future.

The isolation of the subjects of the old curricula has produced sterility.

"Education has not been able to keep abreast with the innumerable problems of our complex life very largely, because its emperical methods are obsolete."

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* Extracted from a paper read by Mr. E. H. Koch Jr., of New York before a Mathematical Association.

" Even scientific men are recognizing the fact that they must apply scientific methods to determine the circumstances which promote or hinder the advancement of science."*

It is surely time for educators to apply the scientific method to determine those factors which enter into the problem of the efficiency of educational systems and more particularly to the special problem regarding the training for efficiency in elementary mathematics.

The best definition of efficiency is borrowed from the technical man who first ascribed it in a commercial way to a machine. Efficiency is the ratio of net output to gross input.

(*a*) $$\text{Efficiency} = \frac{\text{output}}{\text{input}}.$$

The human being is also a machine in the sense that it is a highly organized complexity of more or less co-ordinated parts. The intellectual attributes when associated with the nervous system bear to the framework of the body the analogous relation that the forces of nature bear to structural material.

Efficiency is a measure and is therefore expressed as a ratio between a realized performance and an ideal possibility. There are as many different aspects of efficiency as there are considerations in any mental or physical situation. When you speak of health or strength you unconsciously give efficiency a special name.

It was quite natural to expect technical men to apply this term to the situation which presents itself in technical and industrial education, because the facts in their case were more definitely known. In this respect they have made a beginning.

It is my purpose today to indicate how such an application can be made in the special field of mathematics. Let us consider the summary of our technical colleagues who volunteer the following meanings of efficiency applied to education* :—

1. The efficiency of the curriculum—the ratio of the useful information and mental training offered by a college to that which an ideal college should offer.

(I) E_c = Efficiency of curriculum

$$= \frac{\text{what is offered}}{\text{what should be offered}},$$

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* Science November 4th 1910.

* Karapetoff: *Contributions to the Discussion of Efficiency in Engineering Education*, S.P.E.E. Bulletin, Vol. XVIII.

2. The efficiency of the teacher—the ratio of the amount of knowledge and mental development acquired by the student to the amount of the opportunities offered by the college.

(2) E_t = Efficiency of the teacher

$$= \frac{\text{what and how he teaches}}{\text{what and how he should teach}},$$

3. Efficiency of the student as receiver—the ratio of that which the student assimilates to that which college offers.

(3) E_r = Efficiency of student as receiver

$$= \frac{\text{what he assimilates}}{\text{what the college offers}},$$

4. Efficiency of the student as giver—the ratio of the use made by the student of his college education during his professional career, to the potential energy stored in him at the time of graduation.

(4) E_g = Efficiency of student as giver

$$= \frac{\text{what he gives to the country}}{\text{what he has received from the college}}.$$

There was a time when the soldier was the most efficient man in the State. Centuries later the mantle of efficiency befell to the lot of the lawyer. The evolution of the political, social and industrial progress in the development of civilization compels the finger of time to point to the men of science as the dominant leaders of the nations.

"In America at least", says Professor Townsend,* "we have come to accept as a fundamental principle that the supreme test of an education is the efficiency of the training it gives the individual to meet the demands of organized society and at the same time enable him to contribute most directly or indirectly to the general progress of national life."

Therefore

(5) E_m = Man's overall efficiency

$$= \frac{\text{his productiveness} + \text{his contributions to society}}{\text{latent possibilities times education in its larger sense}}.$$

$$E_m = \frac{P+C}{LS}, \tag{6}$$

where P = productiveness,

C = contributions to society,

L = Latent possibilities,

S = education in its larger sense.

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* Science, November 4, 1910.

Each one of these quantities is resolvable into innumerable factors many of which although not definitely known are nevertheless familiar bywords in the parlance of educational meetings. The unknown factors are equally important and must be determined and investigated. The task at first thought appears so stupendous as to seem almost insuperable, but if we apply the scientific spirit we need not despair. Some years ago the Government sought to devise an automatic machine for predicting the tides for any locality, and for long stated intervals in advance, The question was carefully undertaken, although the obstacles seemed as insurmountable as the elements entering into weather prognostications. The working equation contained thirty-five variables. Today that machine is a realization. Our problem narrows down when we confine our attention to the elements of mathematics which contribute to the efficiency of the individual.

The individual may be likened to the incandescent lamp which has an efficiency of approximately one per cent. This means a waste of ninety-nine per cent. of the energy stored in the coal. Briefly told the final efficiency is the product of all the contributing efficiencies involved in the mechanical and electrical transformations and transmissions through boiler, engine, generator, wires and filaments of the lamp.

(A) E_s=Efficiency of the system

$$=\frac{\text{energy given out as light.}}{\text{energy stored in the coal}}$$

$$\text{(B)}\quad E_s=\frac{\text{Energy stored in the coal}-\text{total losses of energy}}{\text{energy stored in the coal}}.$$

In like manner, man's overall efficiency may be expressed in terms of losses.

$$\text{(8)}\qquad E_m=\frac{LS-(O+I+F)}{LS}$$

substituting in (6) and (B).

O = lost opportunities,
I = Impediments,
F = Failures.

$$\text{(9)}\quad \therefore\qquad P+C=LS-\{O+I+F\},$$

substituting in (8) and (6).

Or we can represent man's overall efficiency as the product of all the efficiencies in the educational power station.

(10) $E_m=E_c\,E_t\,E_r\,E_g$. See (1) (2) (3) (4).

We are especially interested in the mathematics applied to Ec. We shall define 'mathematics efficiency' as the ratio of the factors which

contribute to man's social, political and industrial abilities to the stimulus and power given by mathematics training.

$$(11) \quad E_m = \text{Mathematics efficiency} = \frac{\text{contributing factors}}{\text{stimulus and power given}}.$$

$$(12) \quad E_m = \frac{X}{M}.$$

$$(13) \quad X = f\,(x_1 x_2 x_3 x_4 \ldots\ldots x_n),$$

$$(14) \quad M = \phi\,(m_1 m_2 m_3 \ldots\ldots m_k),$$

$$(15) \quad \therefore \quad E_m = \frac{f(x_1 x_2 x_3 \ldots\ldots x_n)}{\phi(m_1 m_2 m_3 \ldots\ldots m_k)}.$$

$x_1 x_2 x_3 \ldots\ldots x_n$ are the factors affected by mathematics which contribute to man's social, political and industrial abilities, and X is a function of $x_1 x_2 \ldots\ldots x_n$;

$m_1 m_2 m_3 \ldots\ldots m_4$ are the potential contributions of algebra, geometry, analysis and mechanics, and M is a function of $m_1 m_2 \ldots\ldots m_k$.

There are two ways of increasing E_m, either increase the numerator X, or decrease the denominator M.

The known *m's* are usually summarized thus :—

m_1 = the notions of number, measure, form, element,

m_2 = the focussing of law through symbols and graphics,

m_3 = the visualization and interpretation of law through formulas and graphic representation,

m_4 = the appreciation of co-existences and sequences of phenomena,

m_5 = the projection of the intellect beyond experience,

m_k = not defined.

We shall next turn our attention to the *x's* and challenge their validity. We have been talking about these *x's* for a long time and dare say have invented a few out of mere deference for an educational hobby. We must not forget one of the staunchest and earliest critics, Profesor Perry* who holds that the study of mathematics began because it was useful, it continues because it is useful, and it is valuable to the world because of the usefulness of its results in

x_1 = producing the higher emotions and giving mental pleasure,

x_2 = brain development,

x_3 = producing logical ways of thinking,

x_4 = aid given by mathematical weapons in the study of physical science.

* British Association Meeting, Glasgow, 1901.

x_5=passing examinations,

x_6=giving men mental tools as easy to use as their legs or arms, enabling them to go on with their education,

x_7=teaching a man the importance of thinking things out for himself and so delivering him from the present dreadful yoke of authority,

x_8=making men feel that they know the principles,

x_9=giving acute philosophical minds a logical counsel of perfection altogether charming and satisfying.

"Any subject may be useful as applicable to some special purpose or need of life or it may be useful as affording valuable mental discipline."

The following x's were excerpted from various sources:—

x_{10}=mathematics should promote culture, which means the subjugation of imitative power to the creative so as to make the individual develop centrifugal force, individuality, critical opinion and transform that which is read into conversation and life,†

x_{11}=mathematics should maintain the standards of a liberal education which implies a sense of truth and beauty,

x_{12}=mathematics should promote the identical spirit of science and literature which is to transform something of value from the unknown into the realm of the known,

x_{13}=mathematics should respond to the dominant activities of the nation,

x_{14}=mathematics must justify itself in the acquisition of information of commercial value,

x_{15}=mathematics should provoke intellectual tolerance—give breadth of thought,

x_{16}=mathematics should stimulate enthusiasm.

x_{17}=mathematics should instil courage and sympathy with life,

x_{18}=mathematics should give a realization of the value of human life as a public asset,

x_{19}=mathematics should promote intelligent leadership in matters of public concern,

x_{20}=mathematics should promote economy of time and effort,

x_{21}=mathematics should promote deligence and consciousness.

x_{22}=mathematics should promote strength of character,

x_{23}=mathematics should promote judgment,

† Professor Aston, Columbia University, September 28, 1910.

x_{24} = mathematics should promote fidelity and poise,

x_{25} = mathematics should pronote neatness and accuracy,

x_{26} = mathematics should train in habits of attention,

x_{27} = mathematics should train in system in dealing with material,

x_{28} = mathematics should train in clear and interesting presentation both oral and written,

x_{29} = mathematics should render service in the development of the national resources, in aiding the growth and expansion of its industries and of its commercial power and in the conservation of the resources that constitute the inherited wealth of a people,‡

x_{30} = mathematics should furnish a solid foundation on which the more advanced work of college and technical school may be based,

x_{31} = mathematics should develop an appreciation of the methods and spirit of pure science,

x_{32} = mathematics should lead to precision,

x_n = last but not least, mathematics should promote common sense,

There are many more factors of the numerator which may be mentioned. The critical test must be applied to each x, the so called desirable accomplishment, to see if it rings true or whether it has been coined to defend in a vague way some pet fancy of the mathematics department. It is not generally true that in determining the efficiency of any branch of study these same accomplishments are more or less in advocacy. It seems to me that if this were true we would be wasting effort and time in accomplishing in an inefficient way what may be more efficiently accomplished in another subject.

Beyond the ordinary claims mentioned which have been advanced and reiterated is the potent claim of a very broad consideration. I am convinced that in every department of human investigation there are definite underlying mathematical relations which are either little understood or totally unknown.

‡ See Science and Public Service Address, University of Illinois Biologic Station, July 22, 1910. "American Educational Defects," Science, October 28, 1910. Report of Training of Mathematical Instructors, Bulletin A. Math. Soc., November, 1910.

SHORT NOTES.

On the Integral $\int_0^\infty \frac{\cos rxdx}{(1+x^2)^n}$ and some allied Integrals.

§ 1. The solution of Question 472* by Prof. K. J. Sanjana admits of further generalisation as follows :—

Let I denote $\int_0^\infty \frac{\cos rx}{(1+x^2)^n}dx$; differentiating $2n$ times with regard to r, we get

$$\frac{d^{2n}\,\mathrm{I}}{dr^{2n}}=(-1)^n\int_0^\infty \frac{x^{2n}\cos rx}{(1+x^2)^n}dx;$$

or,
$$(-1)^n.\ \frac{d^{2n}\,\mathrm{I}}{dr^{2n}}=\int_0^\infty \frac{x^{2n}\cos rx}{(1+x^2)^n}dx.$$

Similarly

$$(-1)^{n-1}\frac{d^{2n-2}\,\mathrm{I}}{dr^{2n-2}}=\int_0^\infty \frac{x^{2n-2}\cos rx}{(1+x^2)^n}dx;\ \text{etc.}$$

Proceeding in this way, we shall get

$$(-1)^n\left\{\frac{d^{2n}\,\mathrm{I}}{dr^{2n}}-n_1\frac{d^{2n-2}\,\mathrm{I}}{dr^{2n-2}}+\frac{n_2}{2!}\frac{d^{2n-4}\,\mathrm{I}}{dr^{2n-4}}-\dots\dots\dots+(-1)^n\,\mathrm{I}\right\}$$

$$=\int_0^\infty \frac{\cos rx}{(1+x^2)^n}\{x^{2n}+n_1\,x^{2n-2}+\frac{n_2}{2!}\,x^{2n-4}+\dots\dots\dots1\}\,dx$$

$$=\int_0^\infty \frac{\cos rx}{(1+x^2)^n}\{1+x^2\}^n dx$$

$$=\int_0^\infty \cos rxdx=\mathrm{I}_0,\ \text{(say)}.$$

But I_0 is known to vanish (Todhunter, § 291); hence we have the differential equation

$$(-1)^n\,(D^2-1)^n\,\mathrm{I}=0,$$

to determine I_n, where $D\equiv\frac{d}{dr}$.

Solving this, we get

$$\mathrm{I}=e^r\,(A_0+A_1\,r+A_2\,r^2+\dots A^{n-1}\,r^{n-1})+e^{-r}\,(B_0+B_1\,r+\dots B_{n-1}\,r^{n-1}).$$

* Journal I. M. S., December 1913, pp. 230-31.

Now I is finite throughout the range of integration and remains finite however large r may be; hence the terms in e^r must be absent, or $A_0, A_1, A_2 \ldots\ldots A_{n-1}$.

must be each absolutely zero. Thus

$$I = e^{-r}(B_0 + B_1 r + B_2 r^2 + \ldots\ldots B_{n-1} r^{n-1})$$

and we have now to determine B_0, B_1, B_2, etc.

As $e^r I = B_0 + B_1 r + B_2 r^2 + \ldots\ldots + B_{n-1} r^{n-1}$, we obtain, differentiating both sides k times,

$$e^r \left\{ I + kI' + \frac{k_2}{2!} I'' + \ldots\ldots k \cdot I^{(k-1)} + I^{(k)} \right\}$$

$$= \underline{|k}\, B_k + (k+1)k \ldots\ldots 2 \cdot B_{k-1} r + \ldots (A)$$

where the accents denote differentiation with respect to r.

To get B_k we put $r=0$ on both sides.

Observating that $I = \int_0^\infty \frac{\cos rx}{(1+x^2)^n} dx$, we get when $r=0$,

$$(I)_0 = \int_0^\infty \frac{dx}{(1+x^2)^n}.$$

Again $$I' = \int_0^\infty \frac{-x \sin rx}{(1+x^2)^n} dx, \text{ so that } (I')_0 = 0;$$

similarly $$(I''')_0 = (I''''')_0 = \ldots\ldots = 0.$$

Again $$I'' = -\int_0^\infty \frac{x^2 \cos rx \, dx}{(1+x^2)^n},$$

$$\therefore \quad (I'')_0 = -\int_0^\infty \frac{x^2 \, dx}{(1+x^2)^n}; \text{ etc.}$$

We thus get from (A),

$$\underline{|k}\, B_k = \int_0^\infty \frac{dx}{(1+x^2)^n} - \frac{k_2}{2!} \int_0^\infty \frac{x^2 dx}{(1+x^2)^n} + \frac{k_4}{4!} \int_0^\infty \frac{x^4 dx}{(1+x^2)^n} - \ldots\ldots \text{to}$$

$\frac{1}{2}k+1$ terms or $\frac{1}{2}(k+1)$ terms according as k is even or odd.

Putting $x = \tan\theta$, we deduce

$$\int_0^\infty \frac{dx}{(1+x^2)^n} = \int^{\frac{\pi}{2}} \cos^{2n-2}\theta \, d\theta;$$

$$\int_0^\infty \frac{x^2 \, dx}{(1+x^2)^n} = \int_0^{\frac{\pi}{2}} \sin^2\theta \cos^{2n-4}\theta \, d\theta; \text{ etc.}$$

Thus each of these integrals may be expressed in terms of Gamma functions and B_k may be obtained.

To find a single* expression for the series which gives B_k, we write

$$\lfloor k\, B_k=\int_0^{\frac{\pi}{2}}\left\{\cos^{2n-2}\theta-\frac{k_2}{2\,!}\sin^2\theta\cos^{2n-4}\theta+\frac{k_4}{4\,!}\sin^4\theta\cos^{2n-6}\theta-\ldots\right\}d\theta,$$

$$=\int_0^{\frac{\pi}{2}}\cos^{2n-k-2}\theta\left\{\cos^k\theta-\frac{k^2}{2\,!}\cos^{k-2}\theta\sin\theta+\ldots\right\}d\theta$$

$$=\int_0^{\frac{\pi}{2}}\cos^{2n-k-2}\theta\cos k\,\theta\,d\theta$$

$$=\frac{\pi}{2^{2n-k-1}}\left\{\frac{\lfloor 2n-k-2}{\lfloor n-1\,\lfloor n-k-1}\right\},$$

as $2n-2$ is even (see Edward's *Integral Calculus*, Ex. 19, p. 106).

Hence $$B_k=\frac{\pi}{2^{2n-1}\lfloor n-1}\left\{\frac{2^k\lfloor 2n-k-2}{\lfloor k\,\lfloor n-k-1}\right\}.$$

We thus get finally

$$\int_0^\infty\frac{\cos rx\,dx}{(1+x^2)^n}=\frac{\pi e^{-r}}{2^{2n-1}\lfloor n-1}\sum_{k=0}^{k=n-1}\frac{2^k\lfloor 2n-k-2}{\lfloor k\,\lfloor n-k-1}r^k. \qquad \text{(B)}$$

§ 2. From $I=e^{-r}\{B_0+B_1{}^r+B_2\;r^2+\ldots\ldots B_{n-1}\;r^{n-1}\}$ integrating both sides with regard to r, we get

$$J\equiv\int_0^\infty\frac{\sin rx}{x(1+x^2)^n}dx=-e^{-r}\{B_0+B_1(1+r)+B_2(r^2+2r+2)+\ldots+$$
$$B_{n-1}[r^{n-1}+(n-1)_1r^{n-2}+(n-1)_2r^{n-3}+\ldots\ldots(n-1)_{n-1}]\}$$

To determine the constant we put $r=0$; then $(J)_0=0$; and the constant is equal to $B_0+B_1\lfloor 1+B_2\lfloor 2+\ldots+B_{n-1}\lfloor n-1.$

Thus $$J=\{B_0+B_1\lfloor 1+B_2\lfloor 2+\ldots+B_{n-1}\lfloor n-1\}$$
$$-e^{-r}\{B_0+B_0(1+r)+B_2(r^2+2r+2)+\ldots+$$
$$B_{n-1}[r^{n-1}+(n-1)_1r^{n-2}+\ldots\ldots\lfloor n-1]\}. \qquad \text{(C)}$$

§ 3. To find $\int_0^\infty\frac{\sin^2 rx\,dx}{(1+x^2)^n}\equiv K$ (say), we differentiate K with regard to r; we thus get

$$\frac{dK}{dr}=\int_0^\infty\frac{\sin 2\,rx}{x(1+x^2)n}=\{B_0+B_1\lfloor 1+B_2\lfloor 2+\ldots B_{n-1}\lfloor n-1\}$$
$$-e^{-2r}\{B_0+B_1\;(2r+1)+B_2\;(4r^2+4r+2)+\ldots+B_{n-1}$$
$$[(2r)^{n-1}+(n-1)_1\,(2r)^{n-2}+(n-1)_2\,(2r)^{n-3}+\ldots+\lfloor n-1]\}.$$

* This expression is due to Prof. K. J. Sanjana.

Integrating both sides with regard to r, and determining the constant as before, we get the value of K.

Now,

$$\int e^{-2r}\{(2r)^{n-1}+(n-1)_1(2r)^{n-2}+(n-1)_2(2r)^{n-3}+\ldots\lfloor n-1\}\,dr$$

$$=-\frac{e^{-2r}}{2}\Big[(2r)^{n-1}+(n-1)_1(2r)^{n-2}+(n-1)_2(2r)^{n-3}+\ldots+\lfloor n-1$$
$$+(n-1)_1(2r)^{n-2}+(n-1)_2(2r)^{n-3}+\ldots+\lfloor n-1$$
$$+(n-1)_2(2r)^{n-3}+\ldots+\lfloor n-1$$
$$+\ldots\ldots\ldots\ldots$$
$$\text{etc.}\Big]$$

$$=-\frac{e^{-2r}}{2}\{(2r)^{n-1}+2.(n-1)_1(2r)^{n-2}+3.(n-1)_2(2r)^{n-3}+\ldots+\lfloor n\}.$$

The value of the constant is seen to be

$$-\tfrac{1}{2}\{B_0+B_1\lfloor 2+B_2\lfloor 3+\ldots+B_{n-1}\lfloor n\};$$

thus we find

$$K\equiv\int_0^\infty\frac{\sin^2 rx}{x^2(1+x^2)^n}dx=-\tfrac{1}{2}\{B_0+\lfloor 2\,B_1+\lfloor 3\,B_2+\ldots\lfloor n\,B_{n-1}\}$$

$$+r\{B_0+\lfloor 1\,B_1+\lfloor 2\,B_2+\ldots\lfloor n\,B_{n-1}\}$$

$$+\frac{e^{-2r}}{2}\{B_0+B_1(2r+2)+B_2(4r^2+8r+6)$$

$$+B_3(8r^3+24r^2+36r+24)+\ldots\ldots\ldots\ldots+B_{n-1}[(2r)^{n-1}+2.(n-1)_1(2r)^{n-2}+3.(n-1)_2(2r)^{n-3}+\ldots+\lfloor n\}\;\ldots\ldots\ldots\ldots(D).$$

§ 4. The following particular results may be noticed:

(i) Put $n=1$; then (B) becomes

$$\int_0^\infty\frac{\cos rx\,dx}{(1+x^2)}=\frac{\pi e^{-r}}{2}\qquad\ldots\qquad\ldots\quad(1)$$

Making $r=1, 2$, etc.

$$\int_0^\infty\frac{\cos x}{1+x^2}dx=\frac{\pi}{2e},\int_0^\infty\frac{\cos 2x\,dx}{(1+x^2)}=\frac{\pi}{2e^2},\text{ etc. }\ldots\qquad\ldots\quad(2)$$

As $B_0=\frac{\pi}{2}$, the formula (C) becomes

$$\int_0^\infty\frac{\sin rx\,dx}{x(1+x^2)}=\frac{\pi}{2}(1-e^{-r});\qquad\ldots\qquad\ldots\quad(3)$$

whence, making $r=1, 2$, etc.

$$\int_0^\infty\frac{\sin x}{x(1+x^2)}dx=\frac{\pi}{2}\left(\frac{e-1}{e}\right);$$

$$\int_0^\infty\frac{\sin 2x}{x(1+x^2)}dx=\frac{\pi}{2^2e}(e^2-1);\text{ etc.}\qquad\ldots\qquad\ldots\quad(4)$$

Formula (D) gives

$$\int_0^\infty \frac{\sin^2 rx\,dx}{x^2(1+x^2)}=-\frac{1}{2}\cdot\frac{\pi}{2}+\frac{\pi}{2}r+\frac{e^{-2r}}{2}\cdot\frac{\pi}{2}$$

$$=\frac{\pi}{4}(-1+e^{-2r}+2r)\,; \qquad \ldots \qquad \ldots (5)$$

[Ex. 497, Journal, I. M. S., December 1913.]

Taking $r=1$, 2, etc.

$$\int_0^\infty \frac{\sin^2 x\,dx}{x^2(1+x^2)}=\frac{\pi}{4}(1+e^{-2})=\frac{\pi}{4e^2}(1+e^2)\,;$$

$$\int_0^\infty \frac{\sin^2 2x\,dx}{x^2(1+x^2)}=\frac{\pi}{4e^4}(1+3e^4)\,;\text{ etc.} \qquad \ldots \qquad \ldots (6)$$

(ii) Take $n=2$; (B) now gives

$$\int_0^\infty \frac{\cos rx\,dx}{(1+x^2)^2}=\frac{\pi e^{-r}}{2^2}(\underline{|2}+2r)=\frac{\pi e^{-r}}{4}(1+r) \quad \ldots \qquad \ldots (7)$$

Putting $r=1$, 2, etc.

$$\int_0^\infty \frac{\cos x\,dx}{(1+x^2)^2}=\frac{\pi}{2e},\ \int_0^\infty \frac{\cos 2x\,dx}{(1+x^2)^2}=\frac{3\pi}{4e^2} \qquad \ldots \qquad \ldots (8)$$

As now $B_0=\frac{\pi}{4}$ and $B_1=\frac{\pi}{4}$, we obtain from the formulae (C) and (D),

$$\int_0^\infty \frac{\sin rx}{x(1+x^2)^2}dx=\frac{\pi}{2}-e^{-r}\left(\frac{\pi}{2}+\frac{\pi}{4}r\right)$$

$$=\frac{\pi}{4}\{\,2-e^{-r}(r+2)\,\} \qquad \ldots \qquad \ldots (9)$$

$$\int_0^\infty \frac{\sin^2 rx}{x^2(1+x^2)^2}dx=\frac{\pi}{8}\{\,-3+4r+e^{-2r}(2r+3)\,\} \quad \ldots \qquad \ldots (10)$$

(iii) Take $n=3$; (B) gives

$$\int_0^\infty \frac{\cos rx\,dx}{(1+x^2)^3}=\frac{\pi e^{-r}}{2^5\underline{|2}}\left\{\frac{\underline{|4}}{\underline{|2}}+\frac{2\underline{|3}}{\underline{|1}\,\underline{|1}}r+\frac{2^2\underline{|2}}{\underline{|2}}r^2\right\}$$

$$=\frac{\pi}{16}e^{-r}(3+3r+r^2) \qquad \ldots \qquad \ldots (11)$$

[See solution of Ex. 472, December Journal, 1913.)

Here $B_0=\frac{3\pi}{16}$, $B_1=\frac{3\pi}{16}$, $B_2=\frac{\pi}{16}$, so that from (C) and (D) we obtain

$$\int_0^\infty \frac{\sin rx\,dx}{x(1+x^2)^3}=\frac{\pi}{16}\left\{8-e^{-r}(8+5r+r^2)\right\} \qquad \ldots \qquad \ldots (12)$$

[See Ex. 531, February Journal, 1914.]

Also $\displaystyle\int_0^\infty \frac{\sin^2 rx\,dx}{x^2(1+x^2)^3}=\frac{\pi}{32}\{\,-15+16r+e^{-2r}(15+14r+4r^2)\,\}$ ... (13)

(iv) Similarly, we easily obtain :—

$$\int_0^\infty \frac{\cos rx\,dx}{(1+x^2)^4}=\frac{\pi e^{-r}}{96}(15+15r+6r^2+r^3) \quad \dots \quad \dots(14)$$

$$\int_0^\infty \frac{\sin rx\,dx}{x(1+x^2)^4}=\frac{\pi}{96}\{48-e^{-r}(48+33r+9r^2+r^3)\} \quad \dots(15)$$

$$\int_0^\infty \frac{\sin^2 rx\,dx}{x^2(1+x^2)^4}=\frac{\pi}{192}\{-105+96r+e^{-r}(105+114r+48r^2+8r^3)\} \quad (16)$$

$$\int_0^\infty \frac{\cos rx\,dx}{(1+x^2)^5}=\frac{\pi}{768}e^{-r}(105+105r+45r^2+10r^3+r^4) \quad \dots(17)$$

$$\int_0^\infty \frac{\sin rx\,dx}{x(1+x^2)^5}=\frac{\pi}{768}\{384-e^{-r}(384+279r+87r^2+14r^3+r^4)\} \quad (18)$$

20*th March* 1914. T. P. TRIVEDI.

The Face of the Sky for July and August 1914.

Sidereal time at 8 p.m.

	July.				*August.*		
D.	H.	M.	S.		H.	M.	S.
1	6	58	21	...	9	0	35
8	7	25	57	...	9	28	11
15	7	53	33	...	9	55	47
22	8	21	9	...	10	23	23
29	8	48	45	...	10	50	59

Phases of the Moon.

	July.				*August.*		
	D.	H.	M.		D.	H.	M.
Full Moon ...	7	7	30 P.M.	...	6	6	11 A.M.
Last Quarter ...	15	1	2 „	...	14	6	26 „
New Moon ...	23	8	8 A.M.	...	21	5	56 P.M.
First Quarter ...	30	5	21 „	...	28	10	22 A.M.

Eclipses.

On August 21, there is a partial eclipse of the Sun, partly visible at Madras.

First contact	...	6·5 P.M.
Middle „	...	6·58 „
Last „	...	8·1 „

The Planets.

Mercury is stationary on July 2 when it begins to retrograde. It is in inferior conjunction with the Sun on July 16. It is stationary on July 27 and attains its greatest elongation (19°14′ W) on August 5. It is in superior conjunction with the Sun on August 30. It is in conjunction with the Moon on July 22 and on August 20 and with Neptune on August 10.

Venus is an evening star. It is in conjunction with the Moon on July 26 and August 25 and with Mars on August 6.

Mars, which is approaching conjunction with the Sun, is in conjunction with the Moon on August 24 at 5·56 P.M.

Jupiter is in opposition to the Sun on August 11. It is in conjunction with the Moon on July 10.

Saturn, which is in conjunction with the Sun in the middle of June, is in conjunction with the Moon on July 20 at 8·13 and on August 17.

Uranus is in opposition to the Sun on August 3. It is in conjunction with the Moon on July 9.

Neptune is in conjunction with the Sun on July 21 and with the Moon on July 23 and August 19.

V. Ramesam.

The Late Mr. W. Gallatly.

The *Educational Times*, dated March 2, announces the sad news of the sudden death at Boscombe, from heart failure, of Mr. William Gallatly, M.A. His name must be familiar to readers of our Journal as a valuable contributor in Modern Geometry. " A scholar of Pembroke College, Mr. Gallatly obtained his University training and his M.A. Degree at the University of Cambridge. He was both an author and a compiler of works on mathematical subjects, and last year saw the publication of the second (enlarged and carefully revised) edition of his *Modern Geometry of the Triangle*."

SOLUTIONS.

Question 486.

(A. N. RAGHAVACHAR, M.A.) :—If $\alpha, \beta, \gamma, \delta$ be the roots of $x^4+px^3+qx^2+rx+s=0$, form the equation whose roots are

$$\frac{\beta\gamma(\alpha+\delta)-\alpha\delta(\beta+\gamma)}{\beta+\gamma-\alpha-\delta}, \text{ \&c., \&c.}$$

Additional Solution by D. Krishnamurti.

Let $ax^4+4\,bx^3+6cx^2+4\,dx+e=0$ be the proposed equation. Then taking Ferrari's solution of the biquadratic (Burnside and Panton *Theory of Equations*, Vol. I, § 63), we see that

$$\beta+\gamma=-\frac{2}{a}(b-\text{M}),\ \beta\gamma=\frac{c+2a\theta-\text{N}}{a},$$

$$\alpha+\delta=-\frac{2}{a}(b+\text{M}),\ \alpha\delta=\frac{c+2a\theta+\text{N}}{a}.$$

If ρ be a root of the required equation, we find

$$\rho=\frac{b\text{N}-\text{M}\,(c+2a\theta}{a\text{M}}$$

$$\therefore\qquad a\rho+c+2a\theta=\frac{b\text{N}^2}{\text{MN}}$$

$$=\frac{b(c+2a\theta)^2-abe}{bc+ad+2ab\theta},$$

whence, after simplification,

$$\rho=\frac{cd-be+2ad\theta}{bc-ad+2ab\theta}.$$

The required equation is obtained from the reducing cubic by the above homographic transformation. The actual equation is seen to be with the usual notation

$$[cd-be-\rho(bc-ad)]^3$$
$$-\text{I}[cd-be-\rho(bc-ad)](b\rho-d)^2+2\text{J}(b\rho-d)^3=0.$$

Question 502.

(T. P. TRIVEDI, M.A., L.L.B.) :—Prove that the equation of the sides of a regular polygon of $(2n+1)$ sides may be written in the form

$$\text{P}\sin(2n+1)\alpha+(-1)^n\text{Q}\cos(2n+1)\alpha-\text{R}=0,$$

where

$$\text{P}\equiv y^{2n+1}-\frac{2n+1)_2}{2\,!}y^{2n-1}.x^2+\frac{(2n+1)_4}{4\,!}y^{2n-3}.x^4-.(-1.)^n(2n+1)y.x^{2n},$$

$$Q\equiv x^{2n+1}-\frac{(2n+1)_2}{2\,!}x^{2n-1}y^2+\frac{(2n+1)_4}{4\,!}x^{2n-3}y^4-\ldots\ldots(-1^n)(2n+1)xy^{2n},$$

$$R\equiv nar^{2n}-\frac{n(n^2-1^2)}{3\,!}a^3r^{2n-2}+\frac{n(n^2-1^2)(n^2-3^2)}{5\,!}a^5r^{2n-4}\ldots \quad \text{to } (n+1)$$

terms. [Extension of Question 479.]

Solution by N.B. Pendse.

Let the centre of the polygon be the origin and the perpendicular on one side make an angle α with the initial line; and let a be the radius of the inscribed circle. Then

$$r\cos\left(\alpha+\frac{\pi k}{2n+1}-\theta\right)=a,$$

is the equation representing a side of the polygon for different values of k.

Put $\left(\alpha+\frac{2\pi k}{2n+1}-\theta\right)=\phi$, and we have

$$r\cos\phi=a,$$

and
$$\cos(2n+1)\phi=\cos\{(2n+1)(\alpha-\theta)\}.$$

Expanding $\cos(2n+1)\theta$, $\sin(2n+1)\theta$ in terms of $\cos\theta$, $\sin\theta$ and putting x/r, y/r in place of $\cos\theta$, $\sin\theta$, we find

$$r^{2n+1}.\cos(2n+1)\phi=Q\cos(2n+1)\alpha+(-1^n).P\sin(2n+1)\alpha.$$

Again, expand $\cos(2n+1)\phi$ in terms of $\cos\phi$ and substitute a/r for the latter. We have

$$r^{2n+1}\cos(2n+1)\phi=(-1)^nR.$$

Hence, multiplying by $(-1)^n$ throughout,

$$R=(-1)^nQ\cos(2n+1)\alpha+P\sin(2n+1)\alpha,$$

which is the result stated in the question

Question 503.

(R. N. APTE, M.A., L.L.B.) :—A circle is drawn within an ellipse concentric with the ellipse. A tangent to this circle cuts the ellipse in P,P′. The tangents to the ellipse $(x^2/a^2+y^2/b^2=1)$ at P, P′ meet in T, the normals meet in N. If p is the perpendicular from the centre on TN; θ is the angle between PP′ and TN; $a^2+\lambda$, $a^2+\mu$ are the squares of the transverse semi-axes of the confocal conics through T; prove that for the different positions of PP′

$$\frac{1}{\lambda}+\frac{1}{\mu}\ \propto\ p.\sec\theta.$$

Solution by N. B. Pendse.

Let P, P′ be the points α, β. Then obviously T is

$$\{ a\cos\tfrac{1}{2}(\alpha+\beta)\sec\tfrac{1}{2}(\alpha-\beta),\ b\sin\tfrac{1}{2}(\alpha+\beta)\sec\tfrac{1}{2}(\alpha-\beta) \}.$$

i.e., (al, bm) say.

Also N is

$$\{ lc^2\cos\alpha\cos\beta/a, -mc^2\sin\alpha\sin\beta/b \}.$$

The equation of TN is therefore

$$\frac{x-al}{y-bm}=\frac{b}{a}\frac{\cos\frac{1}{2}(\alpha+\beta)(a^2-c^2\cos\alpha\cos\beta)}{\sin\frac{1}{2}(\alpha+\beta)(b^2+c^2\sin\alpha\sin\beta)}.$$

If d is the distance from the origin to the point of intersection of TN and the perpendicular on PP′, we have $p.\sec\theta=d$;

and
$$\frac{y}{x}=\frac{a}{b}\frac{\sin\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha+\beta)}.$$

$$ad\sin\tfrac{1}{2}(\alpha+\beta)=y.\{ a^2\sin^2\tfrac{1}{2}(\alpha+\beta)+b^2\cos^2\tfrac{1}{2}(\alpha+\beta) \}^{\frac{1}{2}}$$
$$=y.n \text{ say,}$$
$$bd\cos\tfrac{1}{2}(\alpha+\beta)=xn.$$

Hence substituting in TN

$$d=\frac{c^2(b^2\cos\alpha\cos\beta+a^2\sin\alpha\sin\beta)n}{ab\{ b^2-a^2+(a^2-b^2)(\cos\alpha\cos\beta+\sin\alpha\sin\beta) \}\cos\frac{1}{2}(\alpha-\beta)}$$
$$= -\frac{n(b^2\cos\alpha\cos\beta+a^2\sin\alpha\sin\beta)}{ab\{ 1-\cos(\alpha-\beta) \}\cos\frac{1}{2}(\alpha-\beta)}.$$

Now λ, μ are the roots of

$$\frac{a^2\cos^2\frac{1}{2}(\alpha+\beta)}{a^2+k}+\frac{b^2\sin^2\frac{1}{2}(\alpha+\beta)}{b^2+k}=\cos^2\tfrac{1}{2}(\alpha+\beta),$$

and the sum of the reciprocals of the roots is therefore

$$\frac{a^2\cos^2\frac{1}{2}(\alpha-\beta)-a^2\cos^2\frac{1}{2}(\alpha+\beta)+b^2\cos^2\frac{1}{2}(\alpha-\beta)-b^2\sin^2\frac{1}{2}(\alpha+\beta)}{a^2b^2\sin^2\frac{1}{2}(\alpha-\beta)}$$
$$=\frac{a^2\sin\alpha\sin\beta+b^2\cos\alpha\cos\beta}{a^2b^2\sin^2\frac{1}{2}(\alpha-\beta)}.$$

$$\therefore\qquad \frac{1}{\lambda}+\frac{1}{\mu}=-\frac{2d\cos\frac{1}{2}(\alpha-\beta)}{abn}.$$

The condition that PP′ should touch a concentric circle of given radius reduces to $\cos\frac{1}{2}(\alpha-\beta)\propto n$.

Hence

$$\frac{1}{\lambda}+\frac{1}{\mu}\propto d.$$

Question 509.

(K. Appukuttan Erady, M. A.) :—Forces act at the point (f,g,h) within the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, and are represented by the normals from the point to the surface. Show that the resultant acts along the line whose direction cosines are

$$\left(\frac{b^2}{a^2-b^2}+\frac{c^2}{a^2-c^2}+1\right)f, \text{ \&c, \&c,}$$

and find its magnitude.

Solution by R. Srinivasan, M. A., and *M. K. Veeraraghavan.*

If the normal at (x, y, z) passes through $(f,g,h,)$, we have

$$\frac{f-x}{x/a^2}=\frac{g-y}{y/b^2}=\frac{h-z}{z/c^2}.$$

$$\therefore \qquad y=\frac{b^2gx}{(b^2-a^2)x+a^2f} \quad \text{and } z=\frac{c^2hx}{(c^2-a^2)x+a^2f}$$

Since (x, y, z) is on the ellipsoid,

$$\frac{x^2}{a^2}+\frac{b^2g^2x^2}{\{(b^2-a^2)x+a^2f\}^2}+\frac{c^2h^2x^2}{\{(c^2-a^2)x+a^2f\}^2}=1.$$

Simplifying, we get

$$x^6(b^2-a^2)^2(c^2-a^2)^2+2x^5\{a^2f(c^2-a^2)(b^2-a^2)^2$$
$$+a^2f(b^2-a^2)(c^2-a^2)^2\}+\dots\dots\dots=0.$$

$\therefore$ If $x_1x_2\dots\dots x_6$ be the roots,

$$\Sigma x=2a^2f\left(\frac{1}{a^2-b^2}+\frac{1}{a^2-c^2}\right)$$
$$=2f\left(\frac{b^2}{a^2-b^2}+\frac{c^2}{a^2-c^2}+2\right).$$

Now, if (α,β,γ) be the mean centre of the feet of the six normals, the direction cosines of the resultant will be proportional to

$$\alpha-f,\ \beta-g,\ \gamma-h\,;$$

i.e., to $\qquad \frac{1}{6}\Sigma x-f,\ \frac{1}{6}\Sigma y-g,\ \frac{1}{6}\Sigma z-h\,;$

i.e., to $\qquad \frac{1}{3}f\left(\frac{b^2}{a^2-b^2}+\frac{c^2}{a^2-c^2}+2\right)-f$, &c., &c.

i.e., to $\qquad f\left(\frac{b^2}{a^2-b^2}+\frac{c^2}{a^2-c^2}-1\right)$, &c., &c.

The magnitude R of the resultant is given by

$$R^2=(f-\alpha)^2+\dots\dots\dots$$
$$=\frac{1}{9}\left\{f^2\left(\frac{b^2}{a^2-b^2}+\frac{c^2}{a^2-c^2}-1\right)^2+\dots\dots\dots\right\}.$$

Question 513.

(Professors, T. P. TRIVEDI & K. J. SANJANA) :—Find the sum of the series

$$1-\frac{h}{h-k+2}+\frac{h(h-1)}{(h-k+2)(h-k+3)}$$
$$-\frac{h(h-1)(h-2)}{(h-k+2)(h-k+3)(h-k+4)}+\dots\dots\dots$$

where h is a positive integer and k any rational number.

Solution by R. Srinivasan, M.A., and K. Rangachari.

Consider the series

$$\frac{1}{m+1}-\frac{n-1}{(m+1)(m+2)}+\frac{(n-1)(n-2)}{(m+1)(m+2)(m+3)}-\dots\dots$$

This can be easily shewn to be equal to $\frac{1}{m+n}$ (*Vide*: Smith's *Algebra*, page 431, Example 14.)

If in this we put $h-k+1$ for m and h for n, we get

$$\frac{1}{h-k+2}-\frac{h-1}{(h-k+2)(h-k+3)}\dots\dots\dots+=\frac{1}{2h-k+1}.$$

The given series is thus equal to

$$1-h.\frac{1}{2h-k+1}=\frac{h-k+1}{2h-k+1}.$$

Question 515.

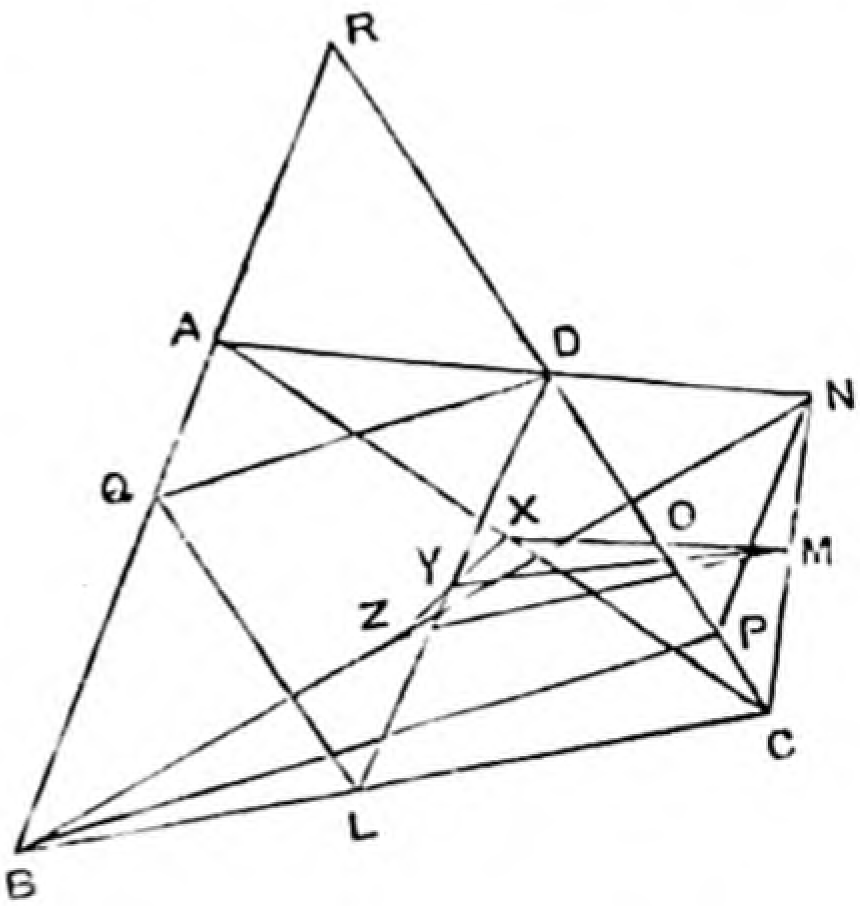
(A. C. L. WILKINSON M.A., F.R.A.S.) :—ABCD is a quadrilateral. DQ, BP are any two parallel straight lines meeting AB, CD respectively in Q and P; QL parallel to CD meets BC in L; and PN parallel to AB meets AD in N. Prove that the middle points of AC, DL, BN are collinear.

Solution by K. Rangachari.

Let BA, CD produced meet in R and let X, Y, Z, M and O be the middle points of AC, DL, BN, CN and CD respectively. Join XM, XY, XZ OY and MZ.

In the ΔACN, X and M are the mid-points of CA and CN. Hence (M passes through O and it is parallel to and a half of AN. Similarly

XO is a half of AD, YO is parallel to and a half of LC, and ZM is parallel to and a half of BC.



In the triangle BRC, LQ is parallel to CR. Hence CL : CB = RQ : RB.

In the triangles RAD, PDN, PN is parallel to AR. Hence PD : DR = ND : DA.

Thus PR : RD = NA : AD = RB : RQ, since in the $\triangle$PRB, QD is parallel to BP.

$$\therefore \quad AN : AD = BC : CL$$

$$\therefore \quad XM : XO = ZM : YO.$$

In the $\triangle$s XOY and XMZ, $X\hat{O}Y = X\hat{M}Z$ and the ratios of the sides about these angles are equal. Therefore $X\hat{Y}O = X\hat{Z}M$.

Hence X, Y, Z are collinear.

Question 516.

(K. J. Sanjana M.A.) :—TP, TQ tangents to a conic of centre C and focus F, cut the auxliary circle in YZ; and FW is perpendicular to the chord of contact; a tangent of the conic perpendicular to TC cuts FY, FZ, FW in y,z,w, respectively. Prove that $yw = wz$ and enunciate the corresponding theorem for the parabola.

Solution by D. Krishnamurti and K. Rangachari.

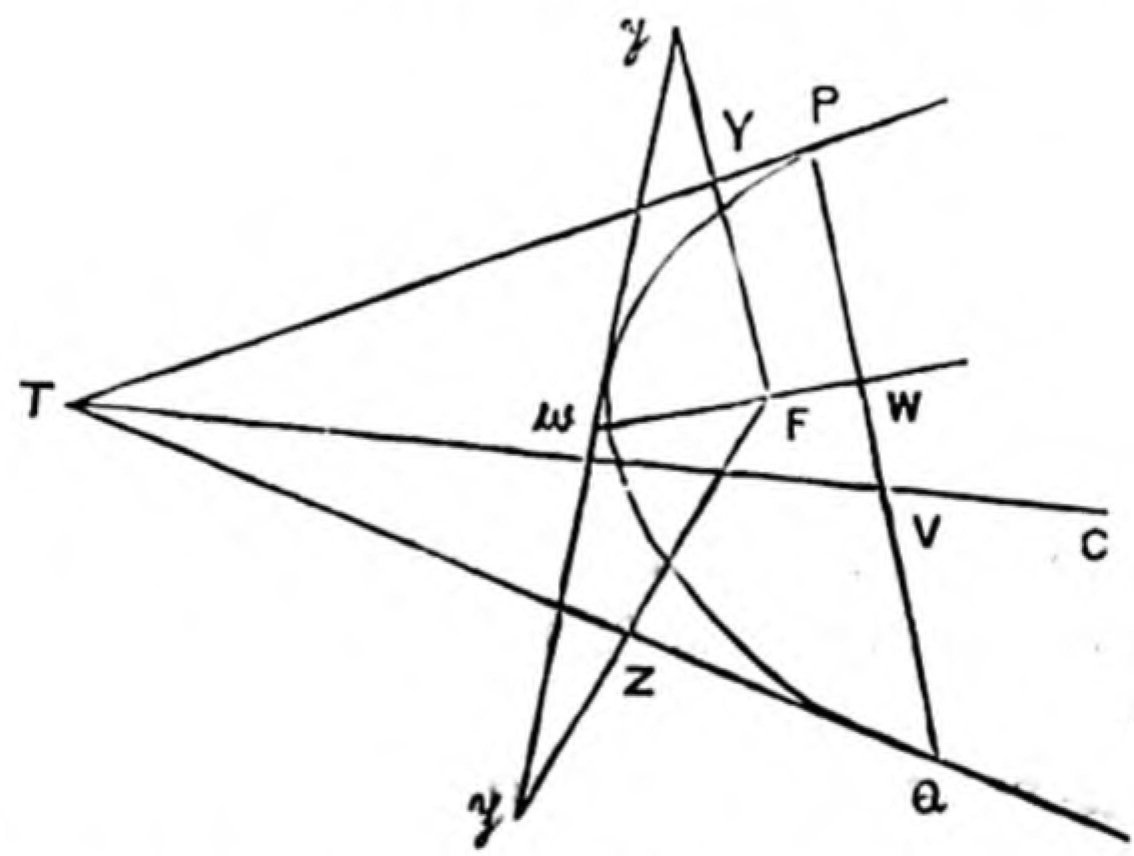
Since Y and Z are points of intersection of the auxiliary circle with the tangents, FY, FZ are ⊥ʳ to TP, TQ. Let CT cut PQ in V.

Comparing △s F*wz* and QVT, we see that they are similar, since F*w*, *wz*, *z* F are respectively ⊥ʳ to QV, VT and TQ.

$$\therefore \qquad wz : TV = Fw : QV \quad \dots \quad \dots \quad \dots (1)$$

In the same way we can show that △ˢ F *wy* and PVT are similar.

$$\therefore \qquad wy : TV = Fw : PV \quad \dots \quad \dots \quad \dots (2)$$



From (1) and (2), since PV = VQ,

$$wz : TU = wy : TU$$

$$\therefore \qquad wz = wy.$$

[*N.B.*—The proposition will hold even if the line *ywz* is not a tangent, provided it is ⊥ʳ to CT.]

In the case of a parabola the proposition will be —

The portion of the tangent at the vertex of a parabola, intercepted by any two tangents, is bisected by the perpendicular from the focus on the chord of contact.

Question 519.

(V. Anantaraman):—Given the base and the vertical angle of a triangle, find the locus of the centre of the circle passing through the three excentres.

Solution by R. Srinivasan, M.A., D. Krishnamurti and K. Rangachari.

We know the circumcircle and the locus of the incentre I (say the circle X on BC). If O, O' be the circumcentres of the triangles ABC, and $I_1I_2I_3$, we know $OO' = IO$; and O is fixed.

$\therefore$ O' divides OI externally in the ratio 1 : 2.

Hence the locus of O' is a curve similar to that of I, and is a circlee

It is easily seen that this circle equals the locus of the incentre and has its centre at the highest point of the circumcircle.

Question 522.

(V. V. S Narayan) :—Given a parallelogram drawn on paper and an ungraduated straight edge, show how to trisect a given finit. straight line on the same paper.

Solution by M. K. Veeraraghavan.

Through the extremities of the given straight line PQ, draw parallels to the sides of the given parallelogram with the ruler.

[For construction see Askwith's *Pure Geometry*, art. 63].

Let PRQS be the parallelogram so formed with PQ for a diagonal. Join RS cutting PQ at C. Through C draw parallels to the sides of the parallelogram, meeting SP, SQ at A and B respectively. Then RA and RB trisect PQ.

The proof is almost obvious.

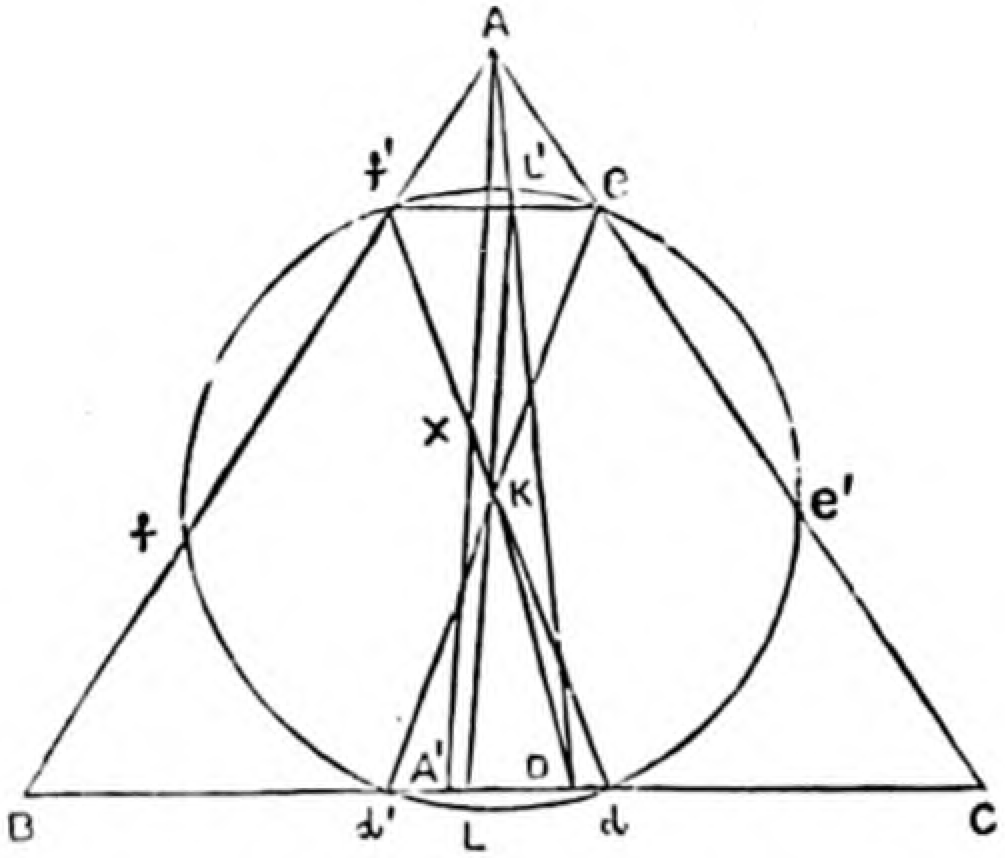
Question 529.

(N. P. Pandya) :—In a triangle ABC, D,E,F are the middle points of BC, CA, AB. If AX, BY, CZ are perpendiculars on the sides of DEF' show geometrically that DX, EY FZ are concurrent at the symmedian point of ABC.

Solution by R. Srinivasan M. A.

Draw the cosine circle cutting BC, CA, AB in d', d; e', e; f', f; then K the symmedian point is its centre. Draw LKL' $\perp^r$ to BC

meeting BC in L and, ef in L′. Then L′ is the mid-point of ef which is $=$ to BC.



∴ AL′ passes through D.

If AA′ be the altitude, X is the mid-point of AA′ and K is the mid-point of LL′.

∴ DKX is a line, since DL′A is a line.

Similarly K lies on EY and FZ.

Question 532.

(R. VYTHYNATHASWAMY) —If 2A, 2B, 2C are the mutual inclinations of any three lines in space and 2L, 2M, 2N the angles which any fourth straight line makes with them, show that the following identical relation holds:

$$\begin{vmatrix} 0, & 1, & 1, & 1, & 1, & 4 \\ 1, & 0, & \sin^2 C, & \sin^2 B, & \sin^2 L, & 1 \\ 1, & \sin^2 C, & 0, & \sin^2 A, & \sin^2 M, & 1 \\ 1, & \sin^2 B, & \sin^2 A, & 0, & \sin^2 N, & 1 \\ 1, & \sin^2 L, & \sin^2 M, & \sin^2 N, & 0, & 1 \\ 4, & 1, & 1, & 1, & 1, & 0 \end{vmatrix} = 0.$$

Solution by R. J. Pocock, B.A., B.Sc., F.R.AS.

Take the three given lines as is coordinate axes and let l, m, n be the ratios of the fourth line. Then

$$\cos 2L = l + m\cos 2C + n\cos 2B$$
$$\cos 2M = l\cos 2C + m + n\cos 2A$$
$$\cos 2N = l\cos 2B + m\cos 2A + n.$$

Multiply by l, m, n and add : $l\cos 2L + m\cos 2M + n\cos 2N = 1$.

We have
$$2(l+m+n-1)-2l-2m-2n+2=0$$
$$l+m+n-1 \qquad -2m\sin^2 C-2n\sin^2 B+2\sin^2 L=0.$$
$$l+m+n-1-2l\sin^2 C \qquad -2n\sin^2 A+2\sin^2 M=0.$$
$$l+m+n-1-2l\sin^2 B-2m\sin^2 A \qquad +2\sin^2 N=0.$$
$$l+m+n-1-2l\sin^2 L-2m\sin^2 M-2n\sin^2 N \qquad =0.$$

From these
$$\begin{vmatrix} 2, & 1, & 1, & 1, & 1, \\ 1, & 0, & \sin^2 C, & \sin^2 B, & \sin^2 L \\ 1, & \sin^2 C, & 0, & \sin^2 A, & \sin^2 M \\ 1, & \sin^2 B, & \sin^2 A, & 0, & \sin^2 N \\ 1, & \sin^2 L, & \sin^2 M, & \sin^2 N, & 0 \end{vmatrix} = 0.$$

Multiplying the 1st row by 2 and bordering the determinant, we obtain

$$\begin{vmatrix} 4, & 2, & 2, & 2, & 2, & 0 \\ 1, & 0, & \sin^2 C, & \sin^2 B, & \sin^2 L, & 0 \\ 1, & \sin^2 C, & 0, & \sin^2 A, & \sin^2 M, & 0 \\ 1, & \sin^2 B, & \sin^2 A, & 0, & \sin^2 N, & 0 \\ 1, & \sin^2 L, & \sin^2 M, & \sin^2 N, & 0, & 0 \\ 4, & 1, & 1, & 1, & 1, & 4 \end{vmatrix} = 0.$$

Subtracting the 1st column from the sixth and then the sixth row from the first row, we get the required result.

QUESTIONS FOR SOLUTION.

548. (A. NARASINGA RAO) :—If $u_r = \frac{x^r}{1-x^{r}}$, shew that the infinite series $u_1+u_2+u_3+u_5-u_6+u_7-u_{10}\ldots = x\frac{(1+x)}{(1-x)}$, where the suffixes of u are the numbers having non-repeated prime factors, the terms being taken with a + ve or — ve sign according as the number of such factors is odd or even.

549. (K. J. SANJANA, M.A.) :—If $\sum_n^r \equiv 1^r-2^r+3^r\ldots+(-1)^{n+1}n^r$,

prove $\sum_n^r + \binom{r+1}{1}\sum_{n-1}^r + \binom{r+2}{2}\sum_{n-2}^r + \ldots\ldots = 0$, if r is even and n is an integer equal to or greater than r. Show how the value of the series may be found when r is odd.

550. (K. J. SANJANA) :—At a game of bridge recently played it was found that every player had three cards of each of three suits and four cards of the remaining suit: what is the chance of this distribution of the cards?

551. (T. P. TRIVEDI, M.A., LL.B.) :—Prove that

$$\int\int\int (abcdfghlmn)(xyz_1)^2\, dx\, dy\, dz,$$

taken throughout a tetrahedron bounded by the co-ordinate planes and the plane $x/p+y/q+z/r=1$ is equal to

$$\frac{pqr}{60}\{(ap^2+bq^2+cr^2+fqr+gpr+hpq)+5(lp+mq+nr)+10d\}.$$

532. (S. KRISHNASWAMI AIYANGAR) :—Find the sum of

$$\frac{2}{3}x+\frac{1}{3\cdot5\cdot2}\frac{x^2}{1!}+\frac{1\cdot3}{5\cdot7\cdot2^3}\frac{x^3}{3!}+\frac{1\cdot3\cdot5}{7\cdot9\cdot2^5}\frac{x^4}{3!}+\ldots\ldots$$
$$+\frac{1\cdot3\cdot5\ldots\ldots2n-1}{(2n+1)(2n+3)2^{2n-1}}\cdot\frac{x^{n+1}}{n!}+\ldots$$

553. (S. KRISHNASWAMI AIYANGAR) :—A B C is a triangle such that its sides subtend equal angles at one focus S of an incribed conic. With S as focus four more conics are drawn circumscribing the triangle. Show that—

(i) The points of contact with the sides of the inscribed conic are on the focal radii through the vertices of the triangle.

(ii) The directrix of one of the circum-conics is identical with the directrix of the inscribed conic corresponding to S.

(iii) The directrix of one of the other circum.conics and its tangent through C meet at the point of intersection of AB and the directrix of the in-conic.

(iv) If this point is P_8, then P_8 S C′ is a right angle where C′ is the point of contact of the inscribed conic with the side AB.

554. (T. P. Trivedi, M.A., LL.B.):—If V denote the entire volume of the solid

$$\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n+\left(\frac{z}{c}\right)^n=1,$$

and A the whole area of its trace on the xy-plane, prove that

$$\frac{V}{A}=\frac{4c}{3n}.\quad B\left(\frac{2}{n},\frac{1}{n}\right),$$

and deduce that for the solid

$$\left(\frac{x}{a}\right)^{\frac{2p}{2p+1}}+\left(\frac{y}{b}\right)^{\frac{2p}{2p+1}}+\left(\frac{z}{c}\right)^{\frac{2p}{2p+1}}=1,$$

$$\frac{V}{A}=\frac{8}{9}\,\frac{c(p+1)(2p+1)}{p(2p+3)(4p+3)}.$$ (Extension of Question 523.)

555 (R. Vythynathaswamy):—P is a point on one of a system of curves $f\,(r_1, r_2, \alpha)=0$ where α is a variable parameter. If dn is the distance at the point P between this curve and its consecutive, whose parameter is $\alpha+d\alpha$, shew that

$$\frac{dn}{d\alpha}=\frac{1}{PQ}.\frac{\partial f}{\partial \alpha}$$

PQ being the diagonal of a parallelogram whose adjacent sides are along r_1, r_2 and equal to $\dfrac{\partial f}{\partial r_1}, \dfrac{\partial f}{\partial r_2}$ respectively.

556. (R. Vythynathaswamy):—Shew that in a cissoid the product of the perpendicular from the cusp on a tangent and the length of the tangent between point of contact and the asymptote varies as the ordinate of the point of contact.

557. (Professor Sanjana):—At a point P of a parabola the normal chord PQ is drawn meeting the axis in G and PN is the ordinate to the axis, prove that PQ. AN = 2 SP. PG.

If from P two chords PP_1, PP_2 are drawn normal to the curve at P_1, P_2, prove further that

$$SP_1 . SP_2 = AS \cdot SP, \quad PP_1 . PP_2 = PQ \cdot AN,$$

where S is the focus.

Shew also that the circle $P_1 P_2 S$ passes through the circumcentre of PP_1P_2.

558. (N. P. PANDYA) :—Solve completely

$$y+(9y+x)\frac{dy}{dx}+3x\left(\frac{dy}{dx}\right)^2+x\left(3y+3x\frac{dy}{dx}+x\right)\frac{d^2y}{dx^2}+x^2y\frac{d^3y}{dx^3}=0.$$

559. (N. SANKARA AIYAR, M.A.) :—Evaluate

$$\int_0^\infty y^{n-1}\coth y\,dy,$$

and

$$\int_0^\infty y^{n-1}\operatorname{cosech}^2 y\,dy.$$

(Suggested by Question 465 of Mr. S. Narayana Aiyar, M.A.)

560. (N. SANKARA AIYAR, M.A.) :—Find the volume contained between the two ellipsoids $x^2/a^2+y^2/b^2+z^2/c^2=1$, and $x^2/b^2+z^2/c^2+y^2/a^2=1$.

561. (P. A. SUBRAMANIA IYER, B.A., L.T.) :—Solve completely

$$(1+3x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-4y=0.$$

List of Periodicals Received.

(From 16th March to 15th May 1914.)

1. Acta Mathematica, Vol. 37, No. 2.
2. Annals of Mathematics, Vol. 15, No. 3, March 1914.
3. Astrophysical Journal, Vol. 39, No. 2, March 1914.
4. Bulletin of the American Mathematical Society, Vol. 20, Nos. 6 and 7, March and April 1914.
5. Bulletin des Sciences Mathematiques, Vol. 38, March and April 1914.
6. Crelle's Journal, Vol. 144, Nos. 2 and 3, February and April 1914.
7. Educational Times, March, April and May 1914, (6 copies)
8. L'Education Mathematique, Vol. 16, Nos. 10, 11, 12 and 13.
9. L'Intermediaire des Mathematiciens, Vol. 21, Nos. 1, 2 and 3, January, February and March 1914.
10. Journal de Mathematiques, Elementaires, Vol. 38, Nos. 10, 11, 12 and 13.
11. Mathematical Gazette, Vol. 7, No. 110, March 1914, (4 copies).
12. Mathematische Annalen, Vol. 75, No. 2, April 1914.
13. Mathesis, Vol. 4, February and March 1914.
14. Messenger of Mathematics, Vol. 43, No. 9 January 1914
15. Monthly Notices of the Royal Astronomical Society, Vol. 74, Nos. 3 and 4, January and February 1914.
16. Philosophical Magazine, Vol. 27, Nos. 159 and 160, March & April 1914.
17. Popular Astronomy, Vol. 22, Nos. 3 and 4, March and April 1914. (3 copies).
18. Proceedings of the London Mathematical Society, Vol. 13, No. 2 March 1914.
19. Proceedings of the Royal Society of London, Vols 89 and 90, Nos. 614 and 615, March and April 1914.
20. Quarterly Journal of Mathematics, Vol. 45, No. 2, March 1914.
21. Revue de Mathematiques Speciales, Vol. 24, Nos. 6 and 7, March and April 1914.
22. School Science and Mathematics, Vol. 14, Nos. 3 and 4, March and April 1914, (3 copies).
23. Transactions of the American Mathematical Sosciety, Vol. 14, Nos. 1 to 4, 1913.
24. Transactions of the Cambridge Philosophical Society, Vol. 22, No. 4, April 1914.
25. Transactions of the Royal Society of London, Vols. 214, No. 510.
26. The Tohoku Mathematical Journal, Vol. 4, No. 4, February 1914.
27. Nature, Vol. 92, Nos. 2310, to 2317, February and March 1914.

The Indian Mathematical Society.

(Founded in 1907 for the Advancement of Mathematical Study, and Research in India.)

THE JOURNAL

OF THE

Indian Mathematical Society

Vol. VI.] AUGUST 1914. [No. 4.

EDITED BY

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Hony. Joint Secretary,

WITH THE CO-OPERATION OF

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and others.

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CONTENTS.

The Journal is open to contributions from members as well as subscribers. The editors may also accept contributions from others.

Contributors will be supplied, if so desired, with extra copies of their contributions at net cost.

All contributions should be written legibly on one side only of the paper, and all diagrams should be given in separate slips.

All communications intended for the Journal should be addressed to the Hony. Joint Secretary, M. T. NARANIENGAR, M.A., Mallesvaram, Bangalore.

All business communications regarding the Journal should be addressed to the Hony. Asst. Secretary, P. V. SESHU AIYAR, B.A., L.T., 37, Venkatachala Chetty Street, Triplicane, Madras.

Enquiries by intending members and all other communications may be addressed to the Hony. Joint Secretary, D. D. KAPADIA, M.A., B.Sc., 322, Main Street, Poona.

THE JOURNAL

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Vol. VI.] AUGUST 1914. [No. 4.

PROGRESS REPORT.

The "*American Mathematical Monthly*" is being received, through the kindness of its publishers, in exchange for your Journal, and numbers of that Journal from the beginning of this year are available for issue on application to the Librarian.

2. The following books have been received as presents to the Library from their publishers—

1. *Bombay University Calendar* 1914, Parts I & II ;
2. *Squaring the Circle*—by Dr. E.W. Hobson, Camb. Univ. Press, 1913, 3*s*. net ;
3. *John Napier*—by Dr. E. W. Hobson, Camb. Univ. Press, 1914, 1*s*. 6*d*. net ;
4. *Complex Integration and Cauchy's Theorem*—by G. N. Watson, (Camb. Math. Tracts, No. 15), 3*s*. net, 1914 ;
5. *Dynamics*—by H. Lamb, Camb. Univ. Press, 1914, 10*s*. 6*d*. net.
6. *A History of Japanese Mathematics*—by D.E. Smith and Yoshio Mikami, The Open Court Publishing Co., 1914, 12*s*. net ;
7. *An Introduction to the Infinitesimal Calculus*—by G. W. Caunt, Oxford Univ. Press, 1914, 12*s*.
8. *Les Coordonnees Intrinseques*—(Theorie et Applications) by L. Braude, (Scientia Series, No. 34), Gauthier-Villars, 1914, 2 frs.
9. *Statics*, Part I—by R. C. Fawdry, G. Bell & Sons, 1914, 2*s*. 6*d*. ;
10. *Arithmetic*—by H. Freeman, G. Bell & Sons, 1914, 2*s*. 6*d*.

POONA, 31st July 1914.

D. D. KAPADIA,
Hony. Joint Secretary.

Statement of Accounts of the Indian Mathematical Society for the half-year ended the 30th June 1914.

Receipts—	Rs.	A.	P.	*Expenditure—*	Rs.	A.	P.
Opening Balance	*970	1	9	Books and Journals	268	14	9
Subscriptions from Members	331	8	0	Library	140	4	0
Subscriptions for Journal	80	13	0	Printing Journal	85	13	6
Miscellaneous	0	14	0	Working expenses	75	9	11
				Closing balance*	812	10	7
Total ...	1,383	4	9	Total ...	1,383	4	9

	Rs.	A.	P.
* With the Treasurer	550	15	1
" Secretary	18	3	0
" Assistant Secretary	90	5	6
" " Librarian	153	3	0
Total ...	812	10	7

S. Narayana Aiyar,

Hon. Treasurer.

Madras,
28th July 1914.

On the Bicircular Quartic.

By A. C. L. Wilkinson, M.A., F.R.A.S.

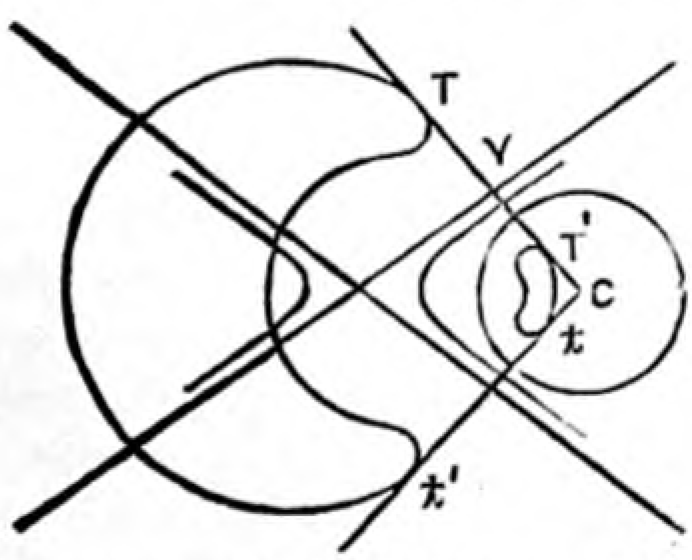
(*Continued from page* 57.)

§ **11.** *On the real forms of bicircular quartics.*

In the fig. on page 50 where the circle of inversion cuts the focal ellipse in four real points, there are four real common tangents to the ellipse and circle $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$, $\delta\delta'$. As the tangent at P turns round from α to δ the point Q moves from α' through Q to δ', and the point Q′ moves from α' through Q′ to δ', where $CQ.\ CQ'=CY^2$. Similarly the portion of the ellipse $\beta\gamma$ gives the second oval of the figure. Between α and β, or γ and δ the tangents to the ellipse intersect the circle, and the corresponding points Q,Q′ are imaginary.

The four points of intersection of the focal ellipse and circle of inversion are circles of zero radius having double contact with the bicircular and are therefore foci of the bicircular.

Every real bicircular has two real double foci, the real common foci of the focal conics, and four other real single foci. (Basset § 79). The positions of these real foci will be indicated in the following discussion.



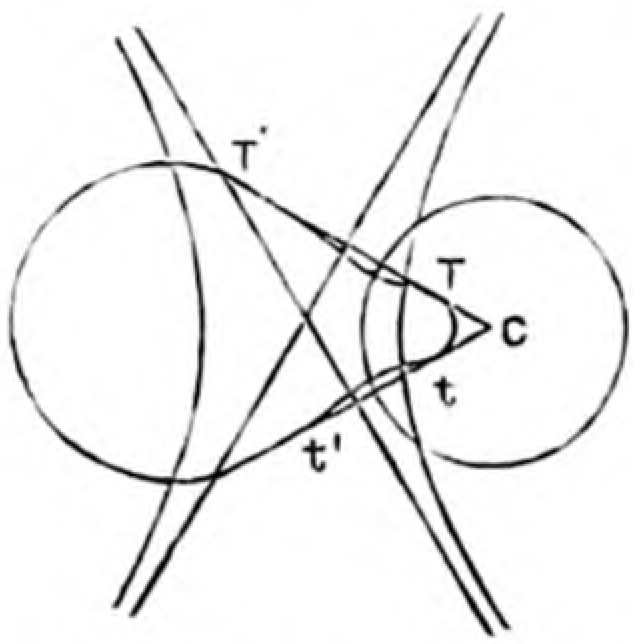
A complete discussion of the real forms of bicirculars can be made by considering the real focal hyperbola and corresponding real circle of inversion. When the circle of inversion does not intersect an asymptote, then the corresponding points of contact of the bitangent are real, as in the fig., where CY is the perpendicular on the asymptote and T, T′ are given by CY^2-YT^2=square of radius of the circle of inversion.

When the circle intersects an asymptote the corresponding bitangent is real but its points of contact with the bicircular are imaginary.

To illustrate the genesis of the complete tables that follow, I shall consider the case of the above fig.

Here we have a circle centre C not intersecting either the curve or the asymptotes. There are two branches of the bicircular, 6 real

bitangents with all the points of contact real, 4 real inflexions, no real foci on the focal hyperbola. As the radius of the circle enlarges, its centre remaining fixed, the points T,T′, t,t′ continually approach each other and when the circle touches the hyperbola the two ovals coalesce into a single curve having a node at the point of contact. When the circle cuts the hyperbola in two real points, but not the asymptote, as in the fig. below, the node disappears and we obtain a single oval with two real bitangents and real points of contact, 4 real inflexions, 2 real foci on the focal hyperbola.

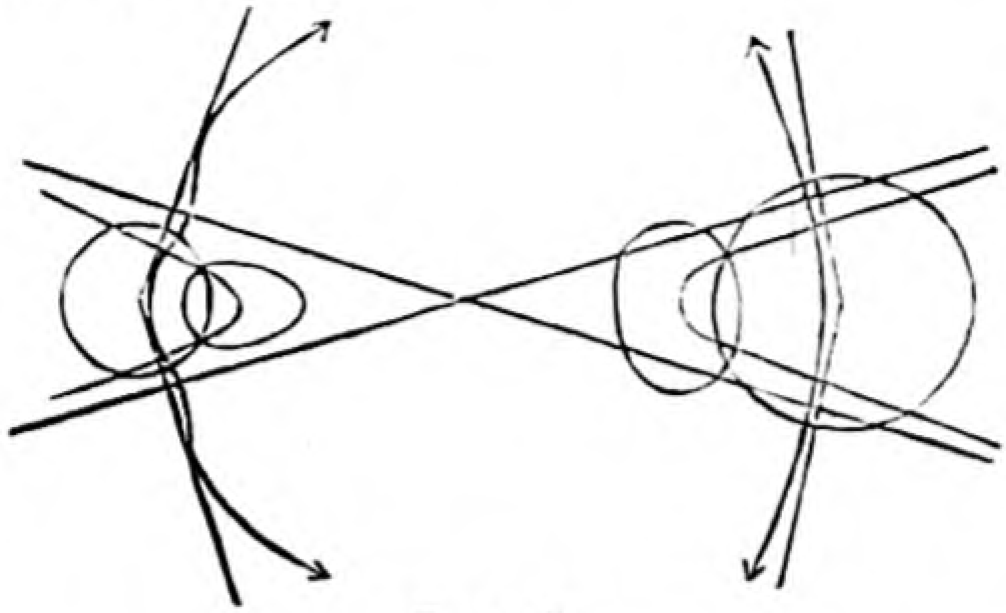


When the circle touches an asymptote the points t,t′ in the fig. coincide and we have a point of undulation on the corresponding bicircular. If the circle cuts one asymptote but not the other the points t,t′ become imaginary, while T,T′ remain real and we obtain a bicircular of one oval, one real bitagent with real points of contact, two real inflexions, two real foci on the focal hyperbola. Continuing to enlarge the circle, the points T,T′ become imaginary, and finally when the the circle touches the other branch of the hyperbola the single oval has shrunk to a conjugate point at the point of contact of the circle and hyperbola. Lastly when the circle intersects both branches of the hyperbola in four real points the bicircular is entirely imaginary.

With this explanation the following table is constructed for the three possible positions of the centre of the circle of inversion with respect to the focal hyperbola:

Notation.—C_0, C_2, C_4 mean none, two, four real intersections of the circle of inversion, in the last case all four must lie on the same branch of the hyperbola. For, when the circle of inversion intersects both branches the corresponding bicircular is imaginary. A_0, A_2, A_4 mean none, two, four real intersections with the asymptotes. Under the column "Bitangents" 5 R. R, 1 R. I, signify five real bitangents with real contacts, one real bitangent with imaginary contacts. Column I

states the real intersections of the circle of inversion with the hyperbol and asympt otes respectively.

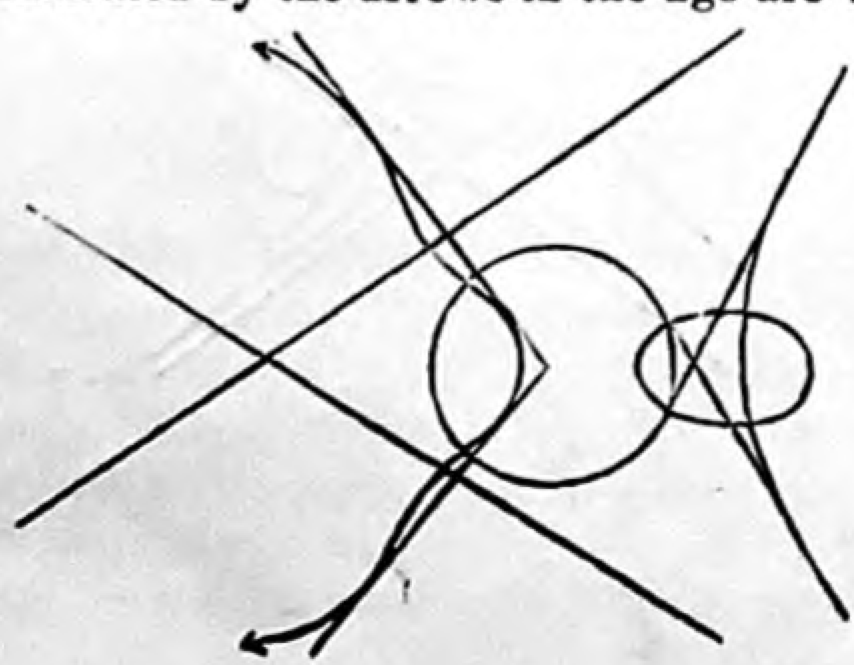


Type I.

The centre of the circle of inversion inside the focal hyperbola.

I	Bicircular.	Bitangents.	Inflexions Real.	Foci on Focal Hyperbola.	Figure.
C_0, A_0	Two exterior ovals.	6. R. R.	4 Two on each oval.	4 Imaginary.	(3)
C_2, A_0	One oval.	2. R. R.	4	2 R.	(4)
C_2, A_2	One oval.	1. R. R. 1. R I.	2	2 R.	...
C_2, A_4	One oval.	2. R. I.	0	2 R.	...
C_4, A_0	Two interior ovals.	2. R. R.	4	4 R.	(5)
C_4, A_2	Two interior ovals.	1. R. R. 1. R. I.	2	4 R.	...
C_4, A_4	Two interior ovals.	2. R. I.	0	4 R.	(6)

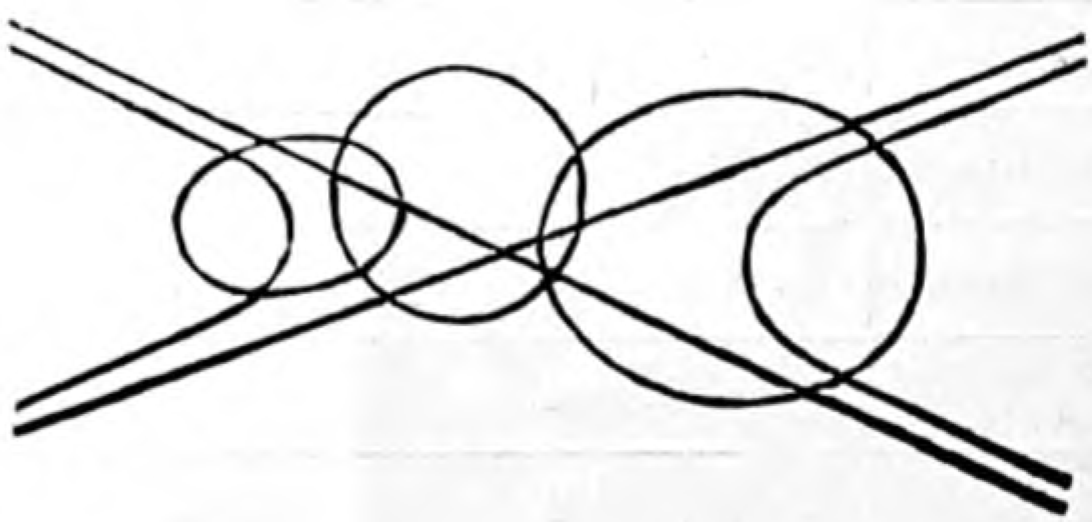
The ovals indicated by the arrows in the figs are very large.

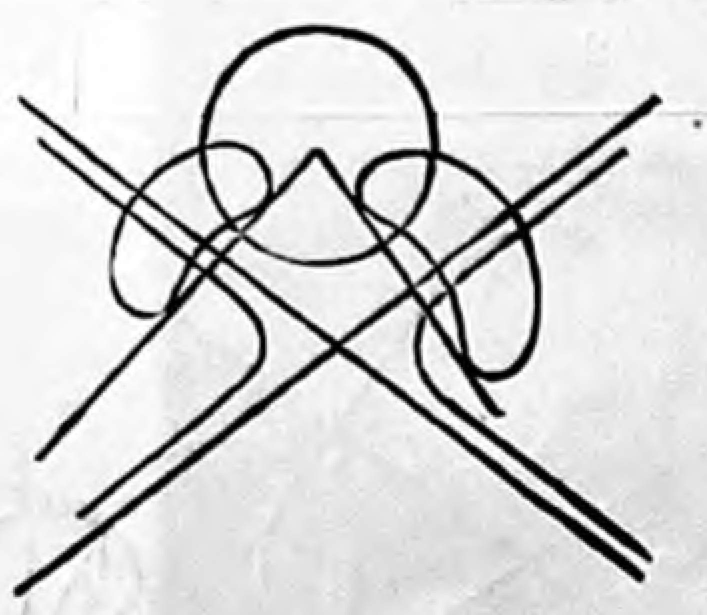


Type II.

Centre of circle of inversion lying between the hyperbola and its asymptotes.

I	Bicircular.	Bitangents.	Inflexions Real.	Foci on focal hyperbola.	Figure.
C_0, A_0	Two exterior ovals.	6 R. R.	4 All on one oval.	4 I	(7)
C_0, A_2	Two exterior ovals.	5 R. R. 1 R. I.	2	4 I	...
C_0, A_4	Two exterior ovals.	4 R. R. 2 R. I.	0	4 I	...
C_2, A_0	One oval.	2 R. R.	4	2 R	...
C_2, A_2	One oval.	1 R. R. 1 R. I.	2	2 R	...
C_2, A_4	One oval.	2 R. I.	0	2 R	...





Type III.

Centre of circle of inversion between the portion of the asymptotes in which the conjugate hyperbola lies.

I	Bicircular.	Bitangents.	Inflexions.	Foci on focal hyperbola.	Figure.
C_0, A_0.	Two exterior ovals.	6 R. R.	4 Nos. on each oval.	4 I.	(8)
C_0, A_2	Two external ovals.	5 R. R. 1 R. I.	2	4 I	...
C_0 A_4	Two external ovals.	4 R. R. 2 R. I.	0	4 I.	(9)
C_2 A_2	One oval.	1 R. R. 1 R. I.	2	2 R.	...
C_2 A_4	One oval.	2 R. I.	0	2 R.	...
C_4 A_2	Two interior ovals.	1 R. R. 1 R. I.	2	4 R.	...
C_4, A_4	Two interior ovals.	2 R. I.	0	4 R.	...

Again since every bicircular can be generated by means of the real focal ellipse and real circle of inversion, we obtain the following classification.

C_0. Two interior ovals with four imaginary foci on the focal ellipse.

C_2. One oval with two real foci on the focal ellipse.

C_4. Two exterior ovals with four real foci on the focal ellipse.

These results, by inspection of figures, will be seen to hold good whether the centre of inversion is inside or outside the ellipse.

The following theorems now follow from the preceeding classification.

1. *When a bicircular consists of two exterior ovals, its four real foci lie on a focal ellipse with a corresponding real circle of inversion.*
2. *When a bicircular consists of two interior ovals, its four real foci lie on a focal hyperbola with a corresponding real circle of inversion.*
3. *When the bicircular consists of a single oval, it has 2 real foci on the focal ellipse with a corresponding real circle of inversion and two real foci on the focal hyperbola with a corresponding real circle of inversion.*

Again reverting to the classification of § 9, we can derive the following results.

4. *When a bicircular can be generated by two real focal ellipses and corresponding real circles of inversion and by one real focal hyperbola and corresponding real circle of inversion, it must consist of two ovals, one interior to the other.*

For were it two external ovals there would be 8 real foci; were it a single oval there would be 2 real foci on each ellipse and 2 real foci on the hyperbola.

5. *When a bicircular can be generated by two real focal hyperbolas and one real focal ellipse with corresponding real circles of inversion, it consists of two ovals mutually exterior to each other.*

6. *When a bicircular can be generated in only two real ways, it consists of a single oval and has two real foci on the real focal ellipse and two real foci on the real focal hyperbola.*

7. *A bicircular quartic whether a single oval or two ovals can have none, two or four real inflexions, but no more.*

§ 12. *The focal properties of the bicircular quartics.*

The following general theorem may be verified. *If from any point on a bicircular quartic tangents are drawn to three circles of the same system having double contact with the bicircular, they are connected by a linear relation.*

Consider the bicircular

$$(x^2+y^2-c)^2-4a^2(x-\alpha)^2-4b^2(y-\beta)^2=0,$$

and three circles

$$x^2+y^2-c-2(x-\alpha)a\cos\theta-2(y-\beta)b\sin\theta=0.$$

Denote these circles by $r_1^2=0$, $r_2^2=0$, $r_3^2=0$ for values θ_1, θ_2, θ_3 respectively of θ.

It is required to verify that quantities l, m, n can be found so that

$$lr_1+mr_2+nr_3=0$$

is identical with the given quartic.

When completely rationalised the linear relation becomes:

$$l^4r_1^4+m^4r_2^4+n^4r_3^4-2l^2m^2r_2^2r_1^2-2l^2n^2r_1^2r_3^2-2m^2n^2r_2^2r_3^2=0.$$

Equating co-efficients between this equation and the given bicircular, the following six relations must be satisfied:

$$\Sigma l^4-2\Sigma l^2m^2=1 \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (1)$$

$$\Sigma l^4\cos\theta_1-\Sigma l^2m^2(\cos\theta_1+\cos\theta_2)=0 \quad \dots \quad \dots \quad \dots \quad (2)$$

$$\Sigma l^4\sin\theta_1-\Sigma l^2m^2(\sin\theta_1+\sin\theta_2)=0 \quad \dots \quad \dots \quad \dots \quad (3)$$

$$2\Sigma l^4\sin\theta_1\cos\theta_1-2\Sigma l^2m^2\sin(\theta_1+\theta_2)=0 \quad \dots \quad \dots \quad (4)$$

$$\Sigma l^4\cos^2\theta_1-2\Sigma l^2m^2\cos\theta_1\cos\theta_2+1=0 \quad \dots \quad \dots \quad \dots \quad (5)$$

$$\Sigma l^4\sin^2\theta_1-2\Sigma l^2m^2\sin\theta_1\sin\theta_2+1=0 \quad \dots \quad \dots \quad \dots \quad (6)$$

From (2) and (3) we deduce

$$\Sigma l^4 e^{i\Theta_1} - \Sigma l^2 m^2 (e^{i\Theta_1} + e^{i\Theta_2}) = 0,$$

which can be written

$$l^2(l^2-m^2-n^2)e^{i\Theta_1} + m^2(m^2-n^2-l^2)e^{i\Theta_2} + n^2(n^2-l^2-m^2)e^{i\Theta_3} = 0,$$

also

$$l^2(l^2-m^2-n^2)e^{-i\Theta_1} + m^2(m^2-n^2-l^2)e^{-i\Theta_2} + n^2(n^2-l^2-m^2)e^{-i\Theta_3} = 0\ ;$$

whence

$$\frac{l^2(l^2-m^2-n^2)}{\sin(\Theta_2-\Theta_3)} = \frac{m^2(m^2-n^2-l^2)}{\sin(\Theta_3-\Theta_1)} = \frac{n^2(n^2-l^2-m^2)}{\sin(\Theta_1-\Theta_2)}$$

By adding the numerators and denominators and taking account of (1), each of these ratios

$$= \frac{1}{-4\sin\frac{\Theta_2-\Theta_3}{2}\sin\frac{\Theta_3-\Theta_1}{2}\sin\frac{\Theta_1-\Theta_2}{2}}.$$

Subtracting (5) and (6), we have

$$\Sigma l^4 \cos 2\Theta_1 - 2\,\Sigma l^2 m^2 \cos(\Theta_1+\Theta_2) = 0.$$

Combining this equation with (4), we obtain

$$\Sigma\, l^4 e^{2i\Theta_1} - 2\,\Sigma\, l^2 m^2 e^{i(\Theta_1+\Theta_2)} = 0.$$

We thus obtain the relations

$$\left.\begin{array}{l} le^{\frac{1}{2}i\Theta_1} + me^{\frac{1}{2}i\Theta_2} + ne^{\frac{1}{2}i\Theta_3} = 0, \\ le^{-\frac{1}{2}i\Theta_1} + me^{-\frac{1}{2}i\Theta_2} + ne^{-\frac{1}{2}i\Theta_3} = 0, \end{array}\right\} \quad \cdots \quad \cdots \quad (7)$$

$$\left.\begin{array}{l} le^{\frac{1}{2}i\Theta_1} + me^{\frac{1}{2}\Theta_2} + ne^{\frac{1}{2}i\Theta_3} = 0, \\ le^{-\frac{1}{2}i\Theta_1} + me^{-\frac{1}{2}i\Theta_2} - ne^{-\frac{1}{2}i\Theta_3} = 0. \end{array}\right\} \quad \cdots \quad \cdots \quad (8)$$

Consider the system (7); solving we get

$$\frac{l}{\sin\frac{\Theta_2-\Theta_3}{2}} = \frac{m}{\sin\frac{\Theta_3-\Theta_1}{2}} = \frac{n}{\sin\frac{\Theta_1-\Theta_2}{2}} = k.$$

Hence

$$l^2(l^2-m^2-n^2) = k^4\sin^2\frac{\Theta_2-\Theta_3}{2}\left\{\sin^2\frac{\Theta_2-\Theta_3}{2} - \sin^2\frac{\Theta_3-\Theta_1}{2} - \sin^2\frac{\Theta_1-\Theta_2}{2}\right\}$$

$$= k^4 \sin(\Theta_2-\Theta_3)\sin\frac{\Theta_2-\Theta_3}{2}\sin\frac{\Theta_3-\Theta_1}{2}\sin\frac{\Theta_1-\Theta_2}{2}.$$

Comparing this with our previous expression, we have

$$k^4 = -\frac{1}{4\sin^2\frac{\Theta_2-\Theta_3}{2}\sin^2\frac{\Theta_3-\Theta_1}{2}\sin^2\frac{\Theta_1-\Theta_2}{2}} \cdots \quad \cdots \quad (10)$$

The values of l, m, n given by (9) and (10) thus render consistent the equations (1), (2), (3), (4) and the difference of (5) and (6). We have finally to show that these values satisfy also the sum of (5) and (6); or that

$$\Sigma l^4-2\,\Sigma l^2 m^2 \cos(\theta_1-\theta_2)+2=0 \quad \ldots \quad \ldots \quad (11)$$

Multiply the two equations of (7) together, therefore

$$l^2+m^2+n^2=-2\left(lm\cos\frac{\theta_1-\theta_2}{2}+mn\cos\frac{\theta_2-\theta_3}{2}+nl\cos\frac{\theta_3-\theta_1}{2}\right)$$

Squaring therefore

$$\Sigma l^4+2\,\Sigma l^2m^2=2\,\Sigma l^2m^2\left\{1+\cos(\theta_1-\theta_2)\right\}$$
$$+8\,lmn\Sigma l\cos\frac{\theta_1-\theta_2}{2}\cos\frac{\theta_3-\theta_1}{2}.$$

Now

$$\Sigma\sin\frac{\theta_2-\theta_3}{2}\cos\frac{\theta_1-\theta_2}{2}\cos\frac{\theta_3-\theta_1}{2}=\sin\frac{\theta_2-\theta_3}{2}\sin\frac{\theta_3-\theta_1}{2}\sin\frac{\theta_1-\theta_2}{2}.$$

Thus

$$8\,lmn\,\Sigma l\cos\frac{\theta_1-\theta_2}{2}\cos\frac{\theta_3-\theta_1}{2}=8\,k^4\sin^2\frac{\theta_2-\theta_3}{2}\sin^2\frac{\theta_3-\theta_1}{2}\sin^2\frac{\theta_1-\theta_2}{2}$$
$$=-2.$$

Hence equation (11) is satisfied.

Considerations of reality of the quantities $\theta_1, \theta_2, \theta_3, l, m, n$ have not so far been taken into account. It would appear at first sight from (10) that all three quantities are not real together; had we however written equations (1), (5), (6) in the form

$$\Sigma l^4-2\,\Sigma l^2 m^2=-1,$$
$$\Sigma l^4\cos^2\theta_1-2\,\Sigma l^2m^2\cos\theta_1\cos\theta_2=1,$$
$$\Sigma l^4\sin^2\theta_1-2\,\Sigma l^2m^2\sin\theta_1\sin\theta_2=1;$$

we should have obtained

$$k^4=\frac{1}{4\sin^2\frac{\theta_2-\theta_3}{2}\sin^2\frac{\theta_3-\theta_1}{2}\sin^2\frac{\theta_1-\theta_2}{2}}.$$

It remains to consider whether such an equation as (8) is possible. Solving (8) we obtain

$$\frac{l}{-\cos\frac{\theta_2-\theta_3}{2}}=\frac{m}{\cos\frac{\theta_3-\theta_1}{2}}=\frac{n}{i\sin\frac{\theta_1-\theta_2}{2}}.$$

The ratio

$$\frac{l^2(l^2-m^2-n^2)}{m^2(m^2-n^2-l^2)} \text{ will be found to be } \frac{\cos^3\frac{\theta_2-\theta_3}{2}\sin\frac{\theta_3-\theta_1}{2}}{\cos^3\frac{\theta_3-\theta_1}{2}\sin\frac{\theta_2-\theta_3}{2}},$$

which is inconsistent with the ratio $\frac{\sin(\theta_2-\theta_3)}{\sin(\theta_3-\theta_1)}$. Thus only equation (7) holds good.

Hence the equation

$$r_1 \sin\frac{\theta_2-\theta_3}{2} \pm r_2 \sin\frac{\theta_3-\theta_1}{2} \pm r_3 \sin\frac{\theta_1-\theta_2}{2}=0$$

when rationalised is identical with the given bicircular quartic.

§ 13. *Application to the real foci of a bicircular.*

Case I: when the bicircular consists of two exterior ovals so that the four real foci lie on an ellipse.

Here $\theta_1, \theta_2, \theta_3$ are real for the point circles or foci and the relation between any three foci and their distances to any point on the bicircular is

$$r_1 \sin\frac{\theta_2-\theta_3}{2} \pm r_2 \sin\frac{\theta_3-\theta_1}{2} \pm r_3 \sin\frac{\theta_1-\theta_2}{2}=0.$$

Case II: when the bicircular consists of two interior ovals, so that the four foci lie on a hyperbola. The four foci all lie on the same branch of the hyperbola and their co-ordinates may be taken to be $(a \cosh \theta_1, b \sinh \theta_1)$ $(a \cosh \theta_2, b \sinh \theta_2)$ $(a \cosh \theta_3, b \sinh \theta_3)$ if they lie on the branch of the hyperbola for which the abscissa is positive.

The relation between the foci and the bicircular is now.

$$r_1 \sinh\frac{\theta_2-\theta_3}{2} \pm r_2 \sinh\frac{\theta_3-\theta_1}{2} \pm r_3 \sinh\frac{\theta_1-\theta_2}{2}=0.$$

Case III: when the bicircular consists of a single oval so that two real foci lie on a focal ellipse and two on a focal hyperbola.

As the circles are not now of the same system no relation of the linear form holds good.

§ 14. *Coversely*, if a bicircular is given by the relation $lr_1+mr_2+nr_3=0$, where r_1, r_2, r_3, are the distances from three fixed points, these points lie on one of the circles of inversion of the bicircular.

Take the circle on which the points lie as

$$(x-\alpha)^2+(y-\beta)^2=\gamma^2.$$

Denote the three points on this circle by

$$(\alpha+\gamma \cos\theta, \beta+\gamma \sin\theta) \text{ for values } \theta_1, \theta_2, \theta_3 \text{ of } \theta.$$

Rationalising the expression

$$\Sigma l \{ (x-\alpha-\gamma \cos\theta_1)^2+(y-\beta-\gamma \sin\theta_1)^2 \}^{\frac{1}{2}}=0,$$

we obtain

$$\Sigma l^4[(x-\alpha)^2+(y-\beta)^2-2\gamma(x-\alpha)\cos\theta_1-2\gamma(y-\beta)\sin\theta_1+\gamma^2]=0.$$

Choose l, m, n so that $\Sigma l^4 - 2\Sigma l^2 m^2 = 1$. Then denoting the expressions (2), (3), (5), (6), (4) of § 12 by A, B, C+1, D+1, E we may write the above bicircular in the form

$$[(x-\alpha)^2+(y-\beta)^2+\gamma^2]^2-4\{(x-\alpha)^2+(y-\beta^2+\gamma^2\}$$
$$\{A\gamma(x-\alpha)+B\gamma(y-\beta)\}$$
$$+4\gamma^2(x-\alpha)^2+4\gamma^2 D(y-\beta)^2+8\gamma^2 E(x-\alpha)(y-\beta)=0;$$

or $$[(x-\alpha)^2+(y-\beta)^2+\gamma^2-2A\gamma(x-\alpha)-2B\gamma(y-\beta)]^2$$
$$=4\gamma^2(A^2-C)(x-\alpha)^2+4\gamma^2(B^2-D)(y-\beta)^2$$
$$+8\gamma^2(AB-E)(x-\alpha)(y-\beta).$$

Now the origin and the direction of the axes are at our disposal. Take $A\gamma+\alpha=0$, $B\gamma+\beta=0$ determining α and β, and choose the direction of the axes so that $AB-E=0$. The bicircular then becomes

$$(x^2+y^2-\alpha^2-\beta^2+\gamma^2)^2=4\gamma^2(A^2-C)(x-\alpha)^2+4\gamma^2(B^2-D)(y-\beta)^2$$

of which the circle of inversion is

$$(x-\alpha)^2+(y-\beta)^2=\gamma^2.$$

§ 15. In certain cases the preceding general theorems degenerate. Suppose $\beta=0$. The corresponding bicirculars are symmetrical. From § 7 we see that one root of the cubic in λ is given by $\lambda+b=0$ and

$$\frac{\beta^2}{b+\lambda}=\frac{c-b-\dfrac{a\alpha^2}{a-b}}{b} \quad \text{in the limit.}$$

The corresponding focal conic degenerates to $g=0$, the value of β' is infinite and the corresponding circle of inversion also degenerates to $g=0$. In this case four of the foci are collinear.

The bicircular now assumes a simple form in § 13.

Consider the focal conic $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ for $b=0$, if three of the foci are at distances u_1, u_2, u_3 from the centre of the focal conics, we have

$$u_1=a\cosh\theta_1,\ u_2=a\cosh\theta_2,\ u_3=a\cosh\theta_3 \text{ and}$$

$$\sqrt{2}\sinh\frac{\theta_1-\theta_2}{2}=\sqrt{\cosh(\theta_1-\theta_2)-1}$$
$$=\frac{1}{a}\sqrt{u_1u_2-a^2-\sqrt{(u_1^2-a^2)(u_2^2-a^2)}}$$
$$=\frac{1}{2a}\left\{\sqrt{(u_1+a)(u_2-a)}-\sqrt{(u_1-a)(u_2+a)}\right\}.$$

If S,S′ are the double foci of the bicircular and F_1, F_2, F_3 three collinear foci, we obtain for the bicircular the equation

$$\sum\left\{\sqrt{SF_1.S'F_2}-\sqrt{S'F_1.SF_2}\right\}r_3=0.$$

Consider further the case of degeneration when the bicircular becomes a cartesian.

Then $a=b$, thus SS' coincide with O the centre of the focal circle.

The ratios of $\sinh\frac{\theta_1-\theta_2}{2} : \sinh\frac{\theta_2-\theta_3}{2} : \sinh\frac{\theta_3-\theta_1}{2}$ have now to be evaluated for $a=0$.

We have

$$\sqrt{2}\sinh\frac{\theta_1-\theta_2}{2}=\frac{1}{a}\sqrt{u_1u_2-a^2-\sqrt{(u_1{}^2-a^2)(u_2{}^2-a^2)}}$$

$$=\frac{1}{a}\sqrt{u_1u_2-a^2-u_1u_2\left(1-\frac{1}{2}\frac{a^2}{u_1{}^2}-\frac{1}{2}\frac{a^2}{u_2{}^2}\right)}.$$

$$=\frac{1}{\sqrt{2}}\left\{\sqrt{\frac{u_1}{u_2}}-\sqrt{\frac{u_2}{u_1}}\right\}$$

Hence if u_1, u_2, u_3 are the distances of the foci of a cartesian from the triple focus, the equation to the cartesian is

$$(u_1-u_2)\sqrt{u_3r_3}\pm(u_2-u_3)\sqrt{u_1r_1}\pm(u_3-u_1)\sqrt{u_2r_2}=0.$$

More generally the above theorem holds good if r_1, r_2, r_3 are the lengths of the tangents from any point on a cartesian to three circles having double contact with the cartesians whose centres lie on the axis of the cartesian at distances u_1, u_2, u_3 from the triple focus.

§ 16. *The fundamental property of a cartesian.*

For a cartesian the focal conics degenerate into circles and in particular one focal hyperbola and the corresponding circle of inversion degenerate into the axis of the cartesian. The equation of a cartesian can be written

$$(x^2+y^2-c)^2=4a^2(x-\alpha)^2+4a^2y^2=0,$$

where the equations of the focal circle and circle of inversion are

$$x^2+y^2=a^2,\ x^2-2\alpha x+y^2+c=0.$$

The cubic in λ becomes

$$(a^2+\lambda)\left\{(a^2+\lambda)(\lambda-c)+a^2\alpha^2\right\}=0.$$

The factor $a^2+\lambda=0$ corresponds to the case where the focal circle and circle of inversion coincide with the axis of x.

The quadratic in λ is

$$\lambda^2+\lambda(a^2-c)+a^2(\lambda^2-c)=0.$$

Now if α^2-c is positive, the circle $x^2+y^2-2\alpha x+c=0$ is real; if α^2-c is negative the roots of the quadratic in λ are real and one is positive and one negative.

From § 9 we now draw the following conclusions:

There are two types of cartesians:

(1) Those for which the cartesian can be generated by means of two real focal circles and corresponding real circles of inversion. These are the ovals of Descartes.

(2) Those for which the cartesian can be generated in only one way by means of a real focal circle and corresponding real circle of inversion.

We may notice that the focal circle and circle of inversion intersect in real points if $2\alpha x-a^2-c=0$ meets $x^2+y^2=a^2$ in real points; that is, if

$$(a^2+c)^2<4\alpha^2a^2,$$

which is the same as the condition that the quadratic in λ should have imaginary roots.

Thus for the first type of cartesians the focal circle and circle of inversion do not intersect in real points and the curve consists of two ovals one within the other. In the second type of cartesians the circles intersect in real points and the curve consists of a single oval.

§ 17. Some of the properties of the focal circle which degenerates to the axis of the cartesian may be obtained as follows:

The cartesian whose equation is

$$(x^2+y^2-c)^2=4a^2(x-\alpha)^2+4a^2y^2$$

can be written

$$(x^2+y^2-c-2a^2)^2+8a^2\alpha x-4a^2(a^2+\alpha^2+c)=0.$$

The straight line $2\alpha x-a^2-\alpha^2-c=0$ is the only bitangent to the cartesian.

From any point on the axis of x a circle can be drawn having double contact with the cartesian. For the circle

$$(x-u)^2+y^2=\rho^2$$

can be written
$$x^2+y^2-c-2a^2=2ux-c-2a^2+\rho^2-u^2,$$
and the condition for double contact is that the equation

$$(2ux-c-2a^2+\rho^2-u^2)^2+8a^2\alpha x-4a^2(a^2+\alpha^2+c)=0$$

may have equal roots.

Now $\qquad (x-b)^2+cx+d=(x+k)^2,$

if $\qquad cx+d=(2x+k-b)(k+b),$

whence $\qquad 2(k+b)=c,\ k-l=2d/c,$

and $\qquad 4b=c-4d/e.$

Thus the condition required is that

$$\frac{c+2a^2-\rho^2+u^2}{u}=\frac{a^2\alpha}{u^2}+\frac{a^2+\alpha^2+c}{\alpha}.$$

This gives ρ^2 for any value of u.

The foci on the axis of x are given by $\rho^2=0$: whence

$$u(u^2+c+2a^2)-a^2\alpha-\frac{a^2+\alpha^2+c}{\alpha}u^2=0;$$

a cubic in u of which one root is $u=\alpha$, and the other roots are given by the quadratic

$$u^2-\frac{u}{\alpha}(c+a^2)+a^2=0.$$

The roots of this are real if $(c+a^2)^2>4a^2\alpha^2$, which is the same as the condition that the roots of the quadratic in λ in § 16 may be real.

Hence when a cartesian consists of two ovals, it has 3 real foci on the axis, when it consists of a single oval it has any one real focus on the axis.

From § 15 if u_1, u_2, u_3 are the distances from the triple focus of any three circles, centres on the axis, having double contact with the cartesian, the equation to the cartesian can be written

$$(u_1-u_2)\sqrt{u_3}\,r_3\pm(u_2-u_3)\sqrt{u_1}\,r_1\pm(u_3-u_1)\sqrt{u_2}\,r_2=0.$$

For the three foci the result of substituting $u_1=\alpha$, u_2 and u_3 the roots of $u^2-\frac{u}{\alpha}(c+a^2)+a^2=0$, gives the cartesian referred to its three collinear foci.

Again, for $u=0$, $\rho=\infty$, and thus the straight line at infinity is a circle having double contact with the cartesian, as is geometrically obvious.

Taking as three circles having double contact with the cartesian, two of the foci and the line at infinity, we have for $u_3=0$, $\rho=\infty$, the limit of $\sqrt{u_3}\,r_3=\sqrt{u_3\,(x^2+y^2-\rho^2)}=a\sqrt{\alpha}$.

Thus the cartesian may be expressed in any one of the four forms

$$(u_1-u_2)\sqrt{\alpha}\,r_3\pm(u_2-\alpha)\sqrt{u_1}\,r_1\pm(\alpha-u_1)\sqrt{u_2}\,r_2=0 \quad \ldots \quad (1)$$
$$a\sqrt{\alpha}\,(u_1-u_2)\pm u_2\sqrt{u_1}\,r_1\pm u_1\sqrt{u_2}\,r_2=0 \ldots \quad \ldots \quad \ldots \quad (2)$$
$$a\sqrt{\alpha}\,(u_1-\alpha)\pm\alpha\sqrt{u_1}\,r_1\pm u_1\sqrt{\alpha}\,r_3=0 \ldots \quad \ldots \quad \ldots \quad (3)$$
$$a\sqrt{\alpha}\,(u_2-\alpha)\pm\alpha\sqrt{u_2}\,r_2\pm u_2\sqrt{\alpha}\,r_3=0 \quad \ldots \quad \ldots \quad (4)$$

where the origin is the triple focus; α, u_1, u_2 the distances of the three foci from the origin; r_3, r_1, r_2 the distances of a point of the cartesian from these foci respectively.

These relations are not independent; for (3) and (4) can be deduced from (1) and (2) by elimination of r_2, r_1 respectively.

§ 18. In conclusion I shall identify the results just obtained with those given in Basset §§ 259 to 277.

The cartesian may be written

$$\{(x+\alpha)^2+y^2-c\}^2=4a^2(x^2+y^2)$$

or $(r^2+2\alpha r\cos\theta+\alpha^2-c)^2=4a^2r^2$, which is identical with Basset's $r+mr_1=k$,

or $$\left\{r^2+\frac{2m^2\lambda}{1-m^2}r\cos\theta+\frac{k^2-m^2\lambda^2}{1-m^2}\right\}^2=\frac{4k^2}{(1-m^2)^2}r^2,$$

where λ, k are written for Basseet's c_1, a_3, if

$$\alpha=\frac{m^2\lambda}{1-m^2},\ \alpha^2-c=\frac{k^2-m^2\lambda^2}{1-m^2},\ a^2=\frac{k^2}{(1-m^2)^2}.$$

Hence $1-m^2=\dfrac{\lambda}{\lambda+\alpha}$, $a^2-\dfrac{\alpha^2-c}{1-m}=\dfrac{m^2\lambda^2}{(1-m^2)^2}=\dfrac{\lambda\alpha}{1-m^2}$, giving for λ the equation

$$\frac{\lambda}{\lambda+\alpha}a^2-(\alpha^2-c)=\lambda\alpha.$$

Writing $\lambda+\alpha=u$, this becomes

$$(u-\alpha)\,a^2-uc=u^2\alpha$$

the quadratic of § 17 giving the foci on the axis of the cartesian other than $u=\alpha$.

Now $u_3=\dfrac{m^2\lambda}{1-m^2}$, $u_1=\lambda+\dfrac{m^2\lambda}{1-m^2}$, $u_2=\lambda'+\dfrac{m^2\lambda}{1-m^2}$, where λ, λ' are the roots of the equation in λ.

The equation (1) of § 17 becomes

$$(\lambda-\lambda')\frac{m\sqrt{\lambda}}{\sqrt{1-m^2}}r_3\pm\lambda'\sqrt{\lambda+\frac{m^2\lambda}{1-m^2}}\,r_1\pm\lambda\sqrt{\lambda'+\frac{m^2\lambda}{1-m^2}}\,r_2=0$$

or $$m(\lambda-\lambda')r_3\pm\lambda' r_1\pm\sqrt{m^2\lambda^2+\lambda\lambda'(1-m^2)}\,r_2=0,$$

which agrees with the result of eliminating the constants between the equations (7) and (8) of Bassett § 262 and substituting for a^2 the expression $m^2\lambda^2+\lambda\lambda'(1-m^2)$ from § 261.

Equation (3) of § 17 becomes

$$\frac{k}{1-m^2}\cdot\frac{m\sqrt{\lambda}}{1-m^2}\lambda\pm\frac{m^2\lambda}{1-m^2}\,\frac{\sqrt{\lambda}}{\sqrt{1-m^2}}r_1\pm\frac{\lambda}{1-m^2}\,\frac{m\sqrt{\lambda}}{\sqrt{1-m^2}}r_2=0,$$

or $$k\pm mr_1\pm r_2=0,$$

agreeing with equation (1) of Bassett § 261.

The other equations can be similarly identified.

Stereographic Projection.

By M. T. Naraniengar.

[Stereographic projection is usually defined as the projection of a spheical curve on the plane of a great circle, the centre of projection being the pole of the great circle. We shall, however, define it as the projection on a tangent plane, the centre of projection being the point diametrically opposite to the point of contact.

Thus defined the stereographic projection of any curve is simply its inverse in space with respect to the origin, the radius of inversion being the diameter of the fundamental sphere].

1. *The stereographic projection of any spherical circle is a circle.*

This follows from the well-known property that the inverse of a circle in space is a circle.

Cor. If the spherical circle passes through the centre of projection the inverse is a straight line. Hence the stereographic projection of such a circle is a straight line.

2. *The angle of intersection of any two spherical curves is unaltered by stereographic projection.*

This is simply a property of inversion.

Hence orthogonal circles on the sphere project into orthogonal circles in a plane, and so on.

3. *The stereographic projection of a diametral spherical triangle ABC* ($C=A+B$) *on the tangent plane at A.*

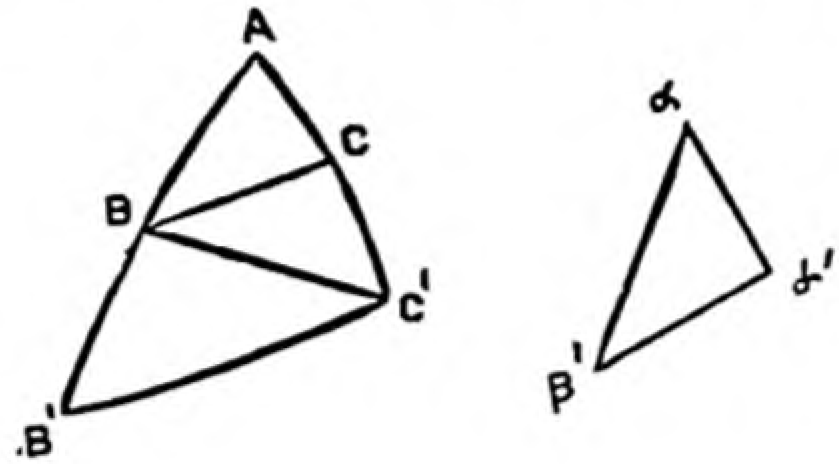
Denoting the projections of A, B, C by the Greek letters α, β, γ ($\alpha \equiv A$) we see that the sides AB, AC, BC of the spherical triangle project into the right lines $\alpha\beta$, $\alpha\gamma$ and an arc $\beta\gamma$. Further, the cicumcircle ABC becomes the circle $\alpha\beta\gamma$; and as AB cuts at right angles the circle ABC, the straight line $\alpha\beta$ must also cut at right angles the circle $\alpha\beta\gamma$. Hence, $\alpha\beta$ is a diameter of the circle $\alpha\beta\gamma$. That is, the plane triangle $\alpha\beta\gamma$ is *right-angled* at γ, having the angle α equal to the angle A.

Cor. The centre of the arc $\beta\gamma$ *is* ω, *such that the angle* $\omega\beta\gamma=\omega\gamma\beta$ $=\pi-C$.

This follows from the property that the arc $\beta\gamma$ makes angles B and C with the right lines $\alpha\beta$, $\alpha\gamma$ respectively.

4. *Properties of a spherical right-angled triangle.*

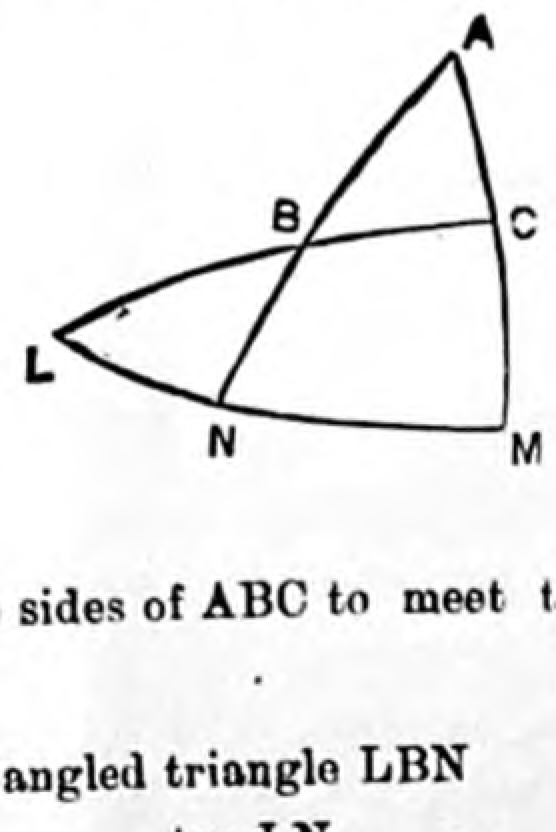
Let ABC be a spherical triangle right-angled at C. Produce AB,



AC to B′, C′ so that BB′ = AB, CC′ = AC. Then evidently $\triangle ABC \equiv C'BC$ and AB′C is a diametral triangle having $C' = A + B'$. Therefore by § 3

$$\cos A = \cos \alpha = \frac{\alpha\gamma'}{\alpha\beta'} = \frac{\tan AOC'}{\tan AOB'} = \frac{\tan \frac{1}{2} AC'}{\tan \frac{1}{2} AB'} = \frac{\tan AC}{\tan AB} = \frac{\tan b}{\tan c}, \ldots(1)$$

O being the point diametrically opposite to A.



Next, produce the sides of ABC to meet the polar great circle of A at L, M, N.

Then in the right-angled triangle LBN

$$\cos L = \frac{\tan LN}{\tan LB} \text{ by (1)}$$

But $L = \frac{\pi}{2} - b$, $LN = \frac{\pi}{2} - A$, $LB = \frac{\pi}{2} - a$, since $LC = LM = 90^\circ$.

Hence $$\sin b = \cot A . \tan a.$$

That is $$\tan A = \frac{\tan a}{\sin b} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Further from the triangles BLN, BAC,

$$\tan B = \frac{\tan LN}{\sin BN} = \frac{\tan AC}{\sin BC}.$$

$$\therefore \qquad \frac{\cot A}{\cos c}=\frac{\tan b}{\sin a}.$$

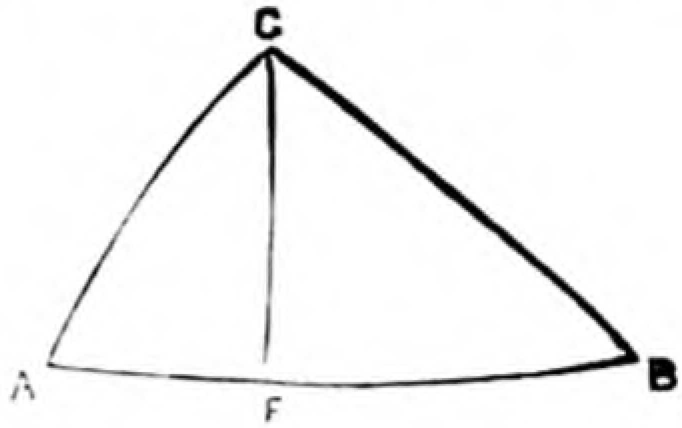
$$\therefore \qquad \cot A=\cos c \tan b \operatorname{cosec} a.$$

Hence, by (2), $\cos c \tan b \operatorname{cosec} a=\cot a \sin b$.

Thus $\qquad \cos c=\cos a \cos b \qquad \ldots \qquad \ldots \qquad (3)$

Using (1), (2) and (3) it is quite easy to deduce all the well-known properties of the right-angled triangle.

5. *Properties of the general spherical triangle.*



These can be made to depend upon the properties of the right-angled triangle established in the previous article. Thus to prove the formula for cos A, we may proceed as follows :

$$\cos A=\tan x \cot b,$$

where x stands for AF, CF being $\perp^r$ to AB.

But $$\cos CF=\frac{\cos b}{\cos x}=\frac{\cos a}{\cos (c-x)}.$$

$$\therefore \qquad \frac{\cos (c-x)}{\cos x}=\frac{\cos a}{\cos b}.$$

$$\therefore \qquad \cos c+\sin c \tan x=\frac{\cos a}{\cos b}.$$

$$\therefore \qquad \tan x=\frac{\cos a-\cos b \cos c}{\sin c \cos b}.$$

Hence

$$\cos A=\tan x \cot b=\frac{\cos a-\cos b \cos c}{\sin b \sin c}.$$

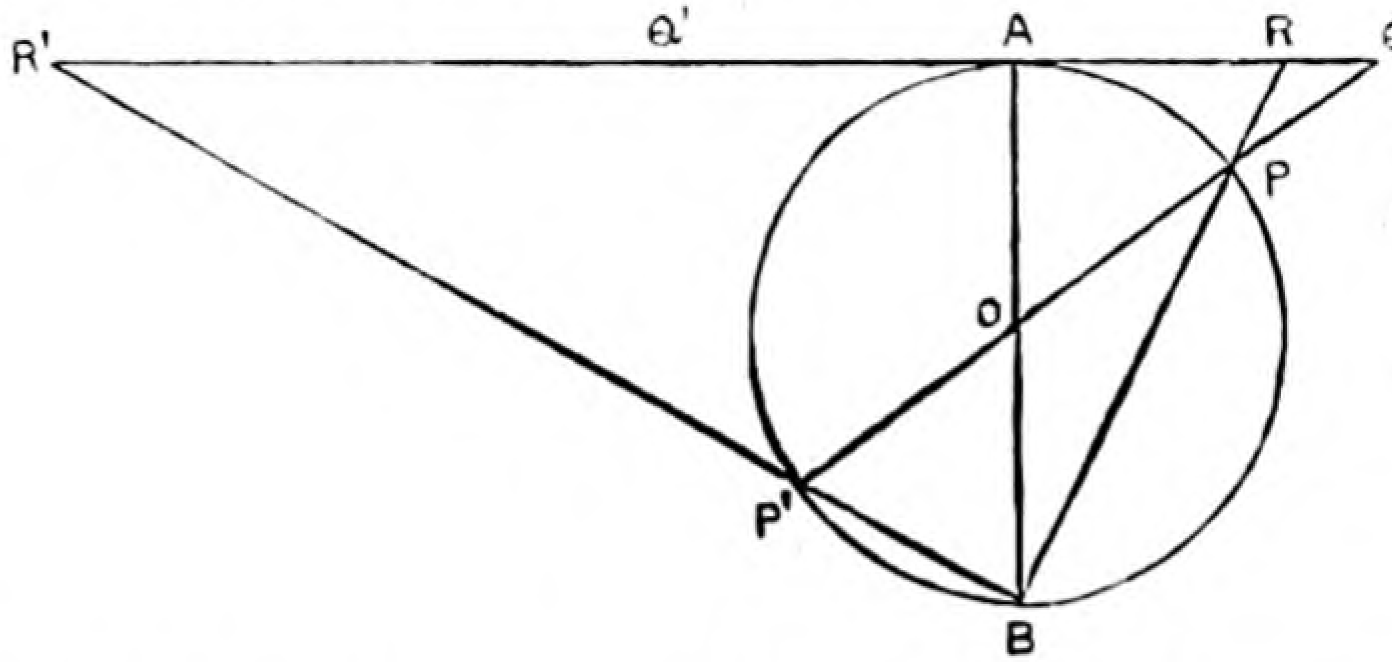
6. *Analytical Investigation.*

Consider the central and stereographic projections of a point P of a sphere on the tangent plane at A. These are Q & R such that ARQ is a straight line. Further, if the angle ABP$=\lambda$,

$$AQ=r=a \tan 2\lambda, \text{ and } AR=r'=2a \tan \lambda.$$

Hence, Q, R are connected by the relation $r=4a^2 r' (4a^2-r'^2)$.

Thus, when the locus of Q is known, that of R is raddily found an is seen to be identical with that of R′, where AR. AR′ $=4a^2$.



The relation between Q & R is geometrically represented as follows :—

$$\frac{4a^2}{r}=\frac{4a^2}{r'}-r'.$$

i.e.
$$\frac{4a^2}{AQ}=\frac{4a^2}{AR}-AR=AR'-AR=2\ AQ'$$

where Q′ is the mid-point of RR′.

That is, $\qquad AQ\ AQ'=2a^2.$

In other words, Q and Q′ describe inverse curves.

Hence, as P describes a spherical curve, we have the following properties among the plane projections connected therewith :—

(1) Q describes the plane projection of the spherical curve.

(2) R, R′ describe the stereographic projection which is such that AR.AR′ $=4a^2$.

Thus the stereographic projection is its own inverse with respect to A, the constant of inversion being $2a\sqrt{-1}$.

(3) The locus of the middle point of RR′ is the inverse of Q with respect to a circle of radius $a\sqrt{-2}$ and centre A.

7. Suppose, the locus of Q is a conic, which may be written, (with the usual notation),

$$u_2+u_1+u_0=0,$$

or
$$r^2f_2(\theta)+rf_1(\theta)+u_0=0.$$

Then the locus of R is obtained by substituting for r, $4a^2r'/(4a^2-r'^2)$ in the above.

The result of the substitution is found to be

$$16a^4r'^2f_2(\theta)+4a^2r'f_1(\theta)(4a^2-r'^2)+u_0(4a^2-r'^2)^2=0,$$

or $$16a^4u_2+4a^2u_1(4a^2-r^2)+u_0(4a^2-r^2)^2=0,$$
if we drop the accents.

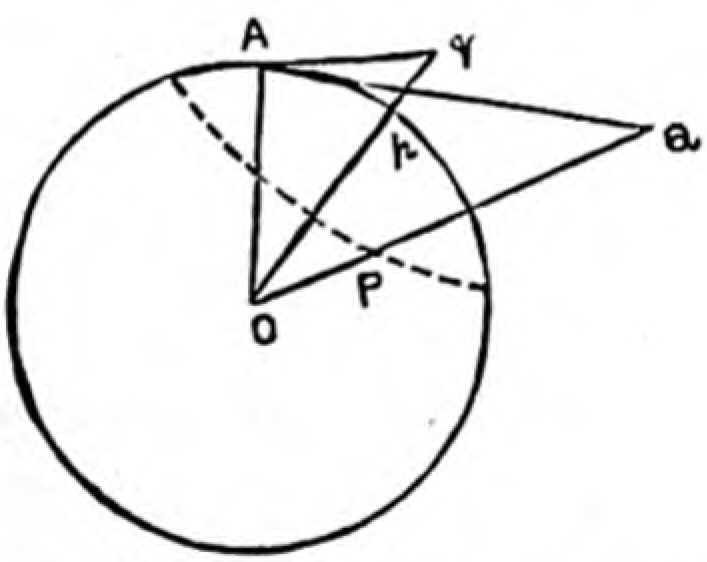
This is a bicircular quartic having the origin for a centre of inversion. (*Vide* : Basset, § 200.)

Casey's method of generation of the quartic therefore shows that Q′ is the foot of the perpendicular from A on a tangent to the focal conic of the quartic traced out by R, R′. In other words, the focal conic is the negative pedal of the locus of Q′, and therefore the locus of Q is the polar reciprocal of the focal conic.

Now, when the locus of Q is a conic, P generally describes a sphero-conic. Hence we have the theorem :

The stereographic projection of a sphero-conic is a bicircular quartic having for a centre of inversion the point diametrically opposite to the centre of projection, whose focal conic is the polar reciprocal of the central projection of the sphero-conic.

In the particular case in which P describes a spherical circle, the stereographic projection degenerates into two circles, and the point A is a centre of inversion for these, being the centre of inverse similitude of the two circles.



Thus, if β be the spherical radius of the circle described by P and p its pole, we have
$$\cos\beta=\cos 2\lambda\cos\alpha+\sin 2\lambda\sin\alpha\cos\theta,$$
where $\alpha=\text{AO}p$, $\theta=\text{QA}q$; and
$$r=a\tan 2\lambda,\ r'=2a\tan\lambda.$$

Hence the locus of R, R′ is
$$\cos\beta\left(+\frac{r'^2}{4a^2}\right)=\pm\left\{\left(1-\frac{r'^2}{4a^2}\right)\cos\alpha+\frac{r'}{a}\sin\alpha\cos\theta\right\},$$
which denotes two inverse circles.

SHORT NOTES.

A card trick.

The explanation of the card trick from the '*Scientific American*' given on p. 15 of Vol. V of the Journal, is as follows:

(Diamonds)	1	2	3	4	5	6	7	8	9	10
(Hearts)	10	1	2	3	4	5	6	7	8	9
(Spades)	9	10	1	2	3	4	5	6	7	8
(Clubs)	8	9	10	1	2	3	4	5	6	7

Denote the position of any card in the above arrangement by (x, y) where x, y are the orders of the column and row to which it belongs counting from the left-hand column and top row respectively. Then it is easily seen that the value of the card is $(x-y+1)$ if this is positive, and $(x-y+1)+10$ otherwise.

In the re-arrangement of the cards, the order of the card (x, y) from 8 of clubs is

$$4\,(x-1)+(5-y)=4\,(x-y+1)+3\,(y-1).$$

Now $(x-y+1)$ is the value of the card if it is positive; and $y-1 =0, 1, 2$ and 3 for diamonds, hearts, spades and clubs respectively.

Hence the rule given.

If $(x-y+1)$ is not positive the rule will give a result in excess of the required number by 40. Hence we should subtract 40, in this case.

R. VYTHYNATHASWAMY.

On Question 155.

[*Q.* 155. (T. Rajarama Rao, B.A., B.L.,) :—BE is the radical axis of the circles ABC and DEF. On EB produced take any point P and draw PAC parallel to the tangent drawn to the circle DEF at the point B cutting the circle BAC at A, C. Similarly, draw PDF parallel to the tangent at B to the circle BAC cutting the circle DEF at D and F. Join AD and CF, and produce them to meet at Q. Shew that PQ is parallel to the line drawn through B such that the intercepts made on the line by the two circles are equal.]

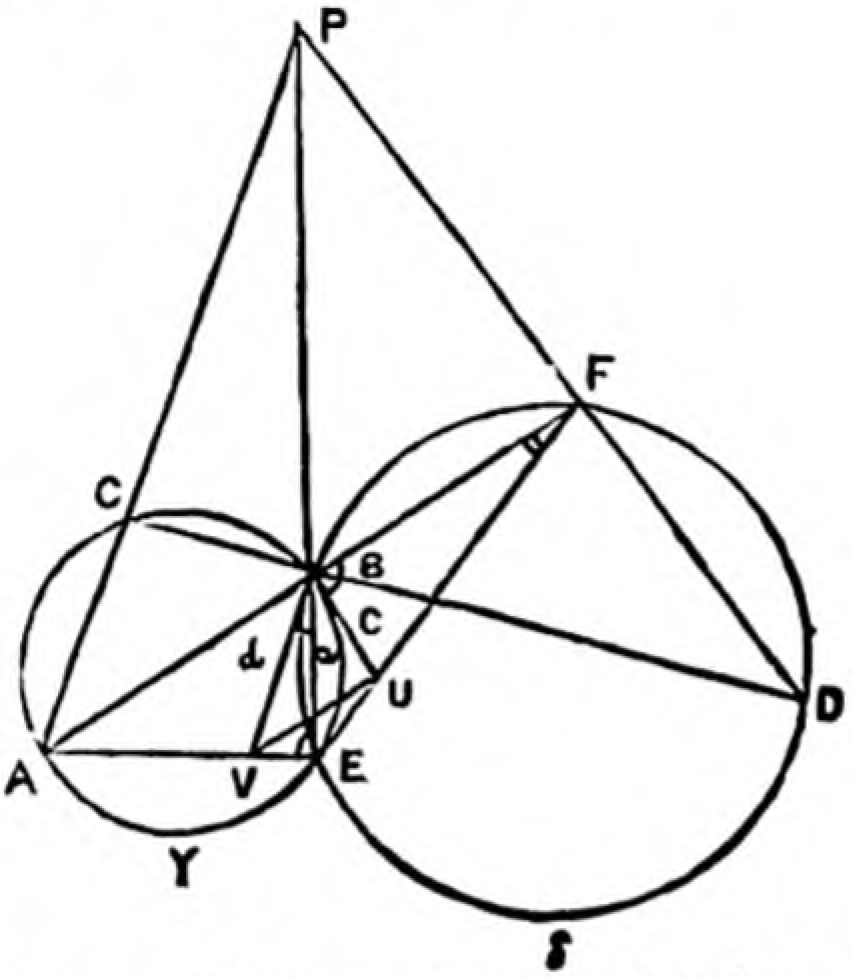
By limiting the position of P to the parts of the radical axis outside the circles, the proposer has been able to present as complete what is in fact only a trivial deduction from two interesting theorems. There are no directions in the enunciation for distinguishing between the points A and C or between the points D and F. The quadrangle ACFD has two harmonic points other than P, and it is only by the position of P that our attention is focussed on the one which has the required property. When we look for the neglected point, we soon find a series of results in which the suggested proposition takes a natural place, for this point is the point B itself. All these theorems are particular cases of theorems on the chords of intersection of any two conics; the analytical treatment of the general case is the simplest way of deducing the results, but we give also geometrical investigations of both the general case and the particular.

1. Two circles γ, δ intersect in B, E and a line through B meets them again in points A, F; if the tangent c to γ at B cuts EF at U and the tangent d to δ at B cuts EA at V, then UV is parallel to AF.

Since $B\hat{E}A = U\hat{B}F$ and $V\hat{B}E = B\hat{F}U$, we have $BV/VE = FU/UB$ similarly $AV/VB = BU/UE$, and the proposition follows on multiplying the ratios.

2. With the same notation as in 1, a line through A parallel to the tangent d, and a line through F parallel to the tangent c intersect on the radical axis BE.

For if the first of these lines meets BE in P, PB/BE = AV/VE = FU/UE and therefore PF is the second line.

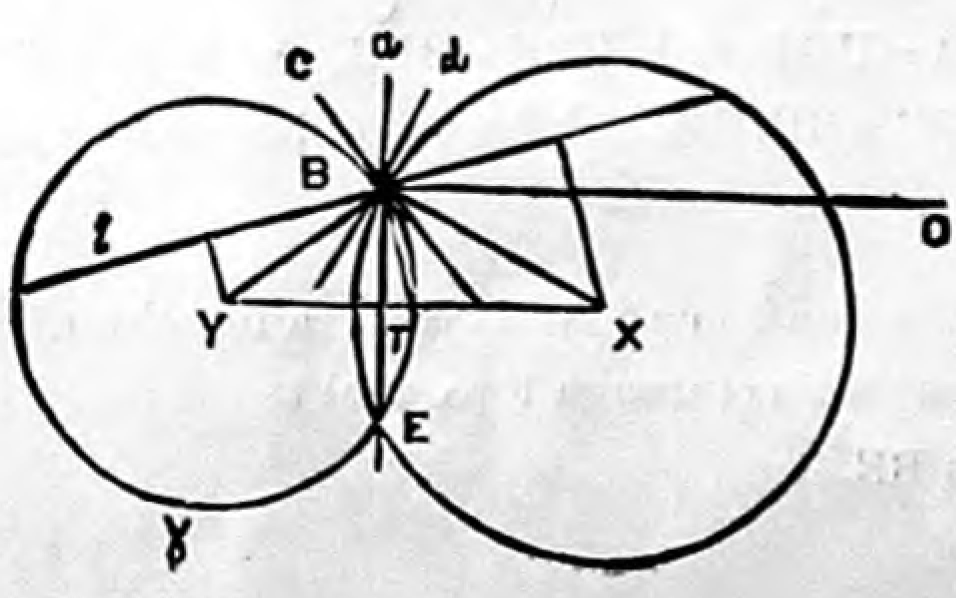


3. Through any point P on the radical axis a of two circles γ, δ of which B is one point of intersection are drawn lines parallel to the tangents c, d at B to γ, δ respectively; if the first of these meets δ in the points D, F and the second meets γ in the points A, C, then B is one of the harmonic points of the quadrangle ACDF.

For if AB, CB meet δ in W, Z the four lines WP, ZP, DP, FP are all parallel to the tangent c, and therefore all coincide, so that since W, Z are distinct points they the same pair of points as D, F.

4. Through a point of intersection B of two circles γ, δ are drawn two pairs of lines, the radical axis a and the line l on which the intercepts made by the two circles are equal forming one pair, and the tangents c, d to the two circles, forming the other; these pairs are harmonic with respect to each other.

Let Y, X be the centres of γ, δ, let T be the middle point of YX, and let O be any point in the line through B parallel to the line of centres YX. Then the lines a, l, c. d are at right angles respectively to BO, BT, BY, BX, and the pair BO, BT is harmonic with respect to the pair BY, BX.



5. Two circles γ, δ have β for a common point ; the chords AC of γ and DF of δ are parallel respectively to the tangents at B to δ, γ, and meet a point P on the radical axis of the circles. Then P and B are two of the harmonic points of the quadrangle ACDF, and the line joining P to the third harmonic point is parallel to the line through B on which the circles make equal intercepts.

This follows at once from what has been proved, since the lines joining P to the other harmonic points from a pair harmonic with the pair PAC, PDF.

6. Let two conics γ, δ meet in the points B, E, L, M, let the tangents at B to γ, δ meet LM in G, H, let any line through B meet γ, δ again in A, F and let EA cut BH in V and EF cut BG in U. Then UV and AF intersect on the line LM, while HA and GF intersect on the line BE.

If EF, EA cut LM in R, S, what we have first to prove is that the ranges EASV, EFRU have the same cross ratio, and we prove this by projecting them both from B on the line LM, that is, we prove that if AF, BE meet LM in K, N, then the cross ratios NKSH, NKRG are the same.

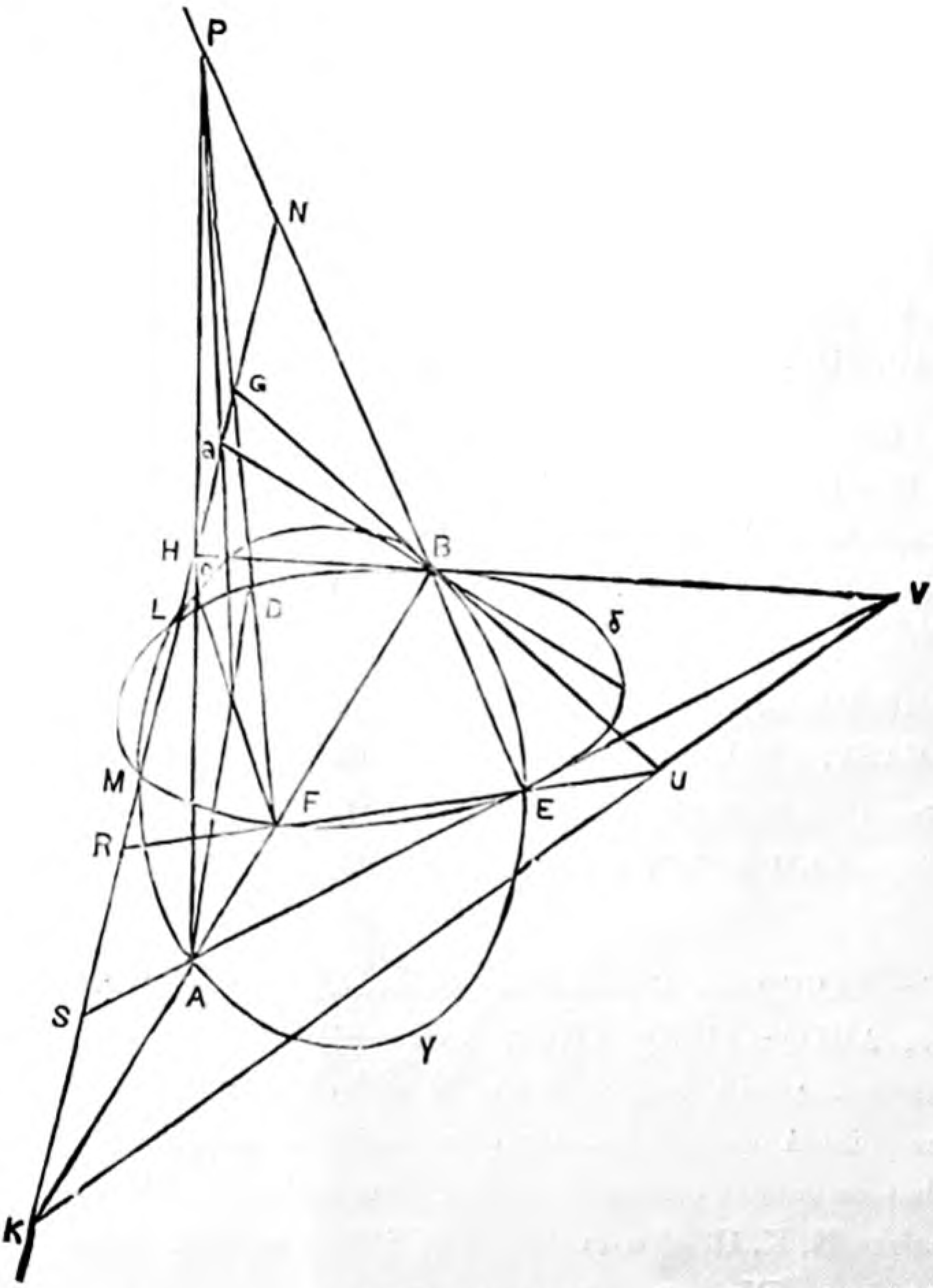
Using ABPQ to denote the cross ratio (AP/PB)$\div$(AQ/QB), we have ABPQ=ABRQ. ABPR. Now projection from the point E on to the conic γ will connect S and N with A and B, and projection from the point A on to the same conic will connect S and K with E and B, but we cannot connect by a single projection S with two of the three points N, K, H. To evaluate NKSH we must therefore express it as a product. We have

NKSH=NKSL. NKLH.

Now since

NLSK=NLSM. NLMK.
=E (BLAM). NLMK.
=B (GLAM). NLMK.
=GLKM. NLMK.
=LGMK. NLMK.
=NGMK.

we have also



$$NKSL = NKMG.$$

Similarly

$$NKLH = NKRM,$$

and therefore

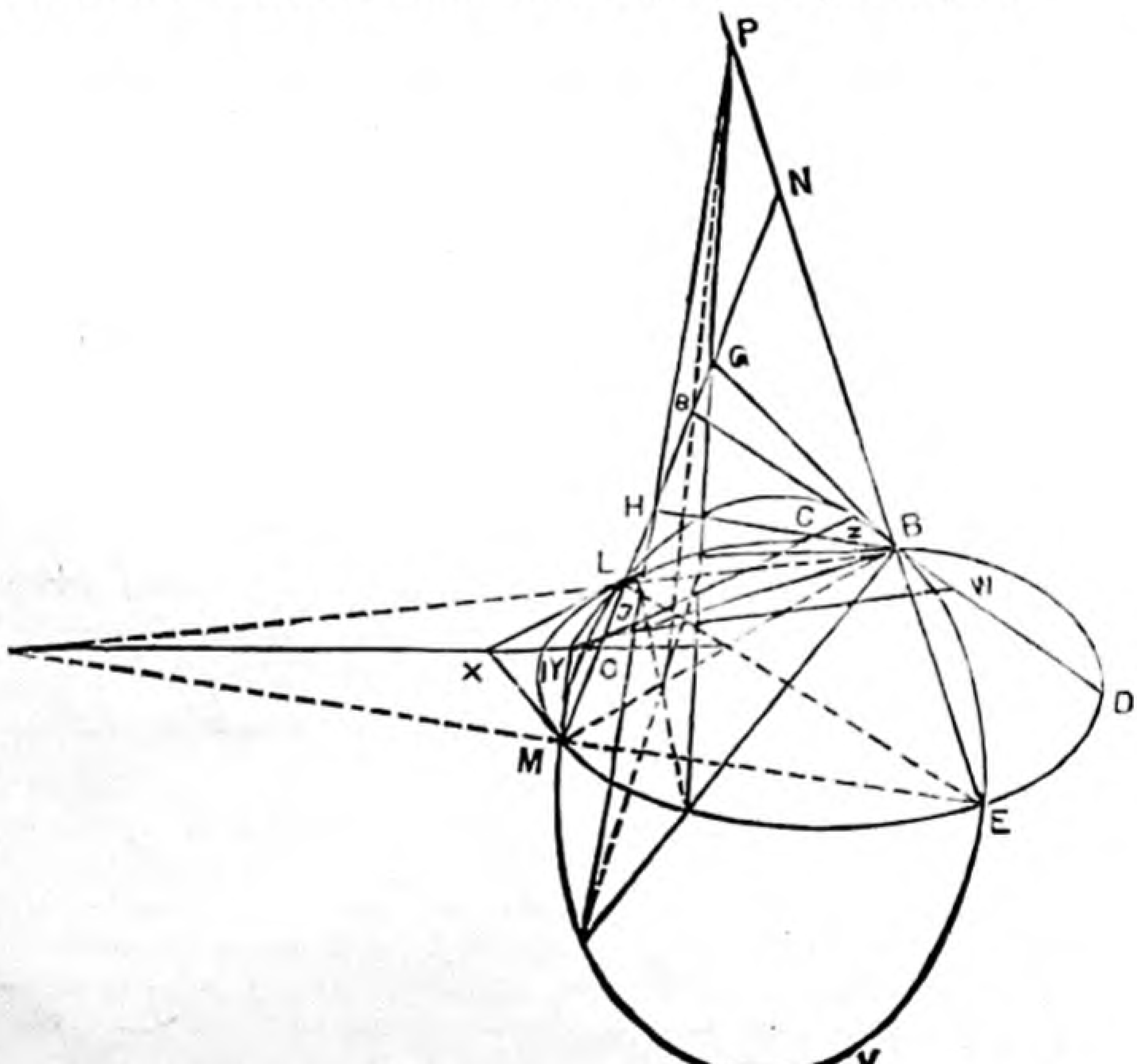
$$NKSH = NKRM.NKMG$$
$$= NKRG.$$

It follows that the ratios EASV, EFRU are equal. Hence UV FA meet on RS, that is, on LM, and also since HE and GE, HS and GR, HV and GU have common points lying on BE, the common point of HA and GF also lies on BE.

7. With the notation of 6, let Q be the point of LM forming with N a pair harmonic with the pair G, H and let P be any point of BE; then the quadrangle formed by the pair of points in which PH meets γ and the pair of points in which PG meets δ has P, B for two of its harmonic points and its third harmonic point lies in PQ.

8. With the same notation, the points distinct form B in which BQ meets the two conics form a pair harmonic with the pair B, Q.

Let J be the point in LM conjugate to Q, and O the point conjugate to N, with respect to the pair of points L, M and let Y, X be the poles of LM for the two conics γ, δ. The polar of N for all conics through B, E, L, M is the line joining the point where BL cuts EM with the points where BM cuts EL, and therefore Y, X, O all lie in this line; let BJ cut this line in I and let YJ, XJ cut QB in Z, W.



Then since the polar of G for γ is BY and the polar of H for δ is BX, the pencils B (LMGY) and B (LMHX) are harmonic; also by hypothesis the pencils B (LMQJ) and B (LMNO) are harmonic. Hence the pencil B (YXJO) has the same ratio as the pencil B (GHQN), that is to say, is a harmonic pencil, and therefore the range YXIO and the pencil J (YXIO) are both harmonic. Thus the pair of points Z, W is harmonic with respect to the pair Q, B.

Again, YJ is the polar of Q for γ and XW is the polar of Q for δ. Hence if BQ meets γ again in C and δ again in D, the ranges QZCB, QWDB are both harmonic, and since QBZW has been proved to be harmonic, it follows that QBCD also is harmonic, which is what we wish to establish.

To reach these conclusions analytically, we may take GHB for our triangle of reference and suppose the conics γ, δ to have the equations

$$ax^2+2hny+by^2+2fyz=0,$$
$$ax^2+2hxy+by^2+2gxz=0,$$

where $ax^2+2hxy+by^2=0$ is the equation of the pair of lines BL, BM. The line BE has for its equation $gx=fy$ and so we may take for any point P in this line the co-ordinates f, g, k. Any point in PG has co-ordinates $f+m, g, k$, for some value of m, any point in PH has co-ordinates $f, g+n, k$, and the equation of the line joining them is

$$k(xn+ym)=(fn+gm+mn)z.$$

Now the first of these points lies on the conic δ if

$$a(f+m)^2+2h(f+m)g+bg^2+2g(f+m)k=0,$$

that is, if

$$a(f+m)^2+2(h+k)(f+m)2+bg^2=0,$$

and similarly the second lies on the conic γ is

$$af^2+2(h+k)f(g+n)+b(g+n)^2=0.$$

From the form of these two equations it follows that if m_1, m_2 are the roots of the first, we may take for the roots of the second n_1, n_2 where

$$(f+m_1)/f=g/(g+n_1),\ (f+m_2)/f=g/(g+n_2),$$

that is, where

$$fn_1+gm_1+m_1n_1=0,\ fn_2+gm_2+m_2n_2=0.$$

Then the lines joining the point $f+m_1, g, k$ to the point $f, g+m_1, k$ and the point $f+m_2, g, k$ to the point $f, g+m_2, k$ have equations

$$xn_1+ym_1=0,\ xn_2+ym_2=0,$$

and both pass through B, which is therefore one harmonic point of the quadrangle formed by these four points.

If the line joining P to the other harmonic point of the quadrangle meets GH in Q, while PB meets GH in N, then PQ, PN are harmonic with respect to PG, PH and so Q, N are harmonic with respect to G, H. But the co-ordinates of N may be taken as f, g, o and therefore the co-ordinates of Q may be taken as $f, -g, o$, and those of any point other than B in BQ as $f, -g, t$. The value of t for this point to lie on γ is given by

$$af^2-2hfg+bg^2-2fgt=0,$$

and the value for the point to lie on δ by

$$af^2-2hfg+bg^2+2fgt=0,$$

and since these values differ only in sign the two points are harmonic with respect to B, Q.

9th April 1914,
Trin. Coll., Cambridge

E. H. NEVILLE.

The Face of the Sky for September and October 1914.

The Sun

enters the autumnal equinox on September 24 at 3·5 A.M.

Phases of the Moon.

	September.			October.		
	D.	H.	M.	D.	H.	M.
Full Moon ...	4	7	31 P.M.	4	11	29 A.M
Last Quarter ...	12	11	18 „	12	3	3 P.M.
New Moon ...	20	3	3 A.M.	19	0	3 P.M.
First Quarter ...	26	5	33 P.M.	26	4	14 A.M.

There will be a partial eclipse of the Moon on September 4 at about 5·16 A.M. and lasting beyond the setting of the Moon.

Planets.

Mercury begins as an evening star. It is in conjunction with the Moon on September 21 and on October 21 and with Mars on October 30.

Venus is an evening star attaining its greatest elongation (46° 27′ E). It is in conjunction with the Moon on September 23 and October 22. It attains its greatest brilliancy on October 23.

Mars is in conjunction with the Moon on September 22.

Jupiter is in conjunction with the Moon on October 27.

Saturn is in quadrature to the Sun on September 26. It is stationary on October 15. It is in conjunction with the Moon on September 13 and October 11.

Uranus is stationary on October 18. It is in conjunction with the Moon on September 2, September 29 and October 26.

Neptune is in quadrature to the Sun on October 24. It is in conjunction with the Moon on September 16 and October 13.

V. RAMESAM.

SOLUTIONS.

Question 297.

(D. D. Kapadia, M.A., B.Sc.):—Prove that the determinant of the n^{th} order

$$\begin{vmatrix} a, & b, & b \ldots b, & b, & b \\ b, & a, & b \ldots b, & b, & b \\ a, & a, & b \,..\, a, & a, & a \\ & \cdots & \cdots & \cdots & \cdots \\ & \cdots & \cdots & \cdots & \cdots \\ a & a & a \ldots b, & a, & a \\ b & b & b \ldots b, & a, & b \\ b & b & b \ldots b, & b, & a \end{vmatrix} = \{\,(n-5)a-3\,b\,\}\,(b-a)^{n-4}$$

where in the first, second, last but one, and last rows there are $(n-1)$ b's and only one b in the other rows.

Solution by N. Sankara Aiyar, M.A.

Subtracting the last column from each of the others we get

$$\Delta = \begin{vmatrix} a-b, & 0, & 0 \ldots 0, & 0, & b \\ 0, & a-b, & 0 \ldots 0, & 0, & b \\ 0, & 0, & b-a, \ldots 0 & 0 & a \\ \cdots & \cdots & \cdots & \cdots & \\ 0, & 0, & 0 \ldots b-a, & 0, & a \\ 0, & 0, & 0 \ldots \;\; 0, & a-b, & b \\ b-a, & b-a, & b-a \ldots b-a, & b-a, & a \end{vmatrix}$$

$$= (b-a)^{n-1} \begin{vmatrix} -1, & 0, & 0, \ldots 0, & 0, & b \\ 0, & -1, & 0, \ldots 0, & 0, & b \\ 0, & 0, & 1, \ldots 0, & 0, & a \\ & \cdots & \cdots & \cdots & \\ 0, & 0, & 0, \ldots 1, & 0, & a \\ 0, & 0, & 0, \ldots 0, & -1, & b \\ 1, & 1, & 1, \ldots 1, & 1, & a \end{vmatrix}$$

$$=(b-a)^{n-1}\begin{vmatrix} -1, & 0, & 0, \ldots 0, & 0, & 0 \\ 0, & -1, & 0, \ldots 0, & 0, & b \\ 0, & 0, & 1, \ldots 0, & 0, & a \\ & \cdots & \cdots & \cdots & \\ 0, & 0, & 0, \ldots 1, & 0, & a \\ 0, & 0, & 0, \ldots 0, & -1, & b \\ 1, & 1, & 1, \ldots 1, & 1, & a+b \end{vmatrix}$$

by multiplying the 1^{st} column by b and adding to the last

$$=-(b-a)^{n-1}\begin{vmatrix} -1, & 0, \ldots & 0, & 0, & b \\ 0, & 1, \ldots & 0, & 0, & a \\ 0, & 0, \ldots & 0, & 0, & a \\ & \cdots & \cdots & \cdots & \\ 0, & 0, \ldots & 0, & -1, & b \\ 1, & 1, \ldots & 1, & 1, & a+b \end{vmatrix}$$

where the determinant is of the $(n-1)^{th}$ order.

Proceeding thus we get

$$\Delta=(b-a)^{n-1}\begin{vmatrix} 1, & 0, & 0, \ldots 0, & 0, & \\ 0, & 1, & 0 \ldots 0, & 0, & a \\ 0, & 0, & 1 \ldots 0, & 0, & a \\ & \cdots & \cdots & \cdots & \\ 0, & 0, & 0 \ldots 1, & 0, & a \\ 0, & 0, & 0 \ldots 0, & -1, & \\ 1 & 1, & 1, & 1, & a+2b \end{vmatrix}$$

$$=(b-a)^{n-1}\begin{vmatrix} -1, & \\ 1, & 2b-(n-5)a \end{vmatrix}$$

$$=(b-a)^{n-1}\{(n-5)a-3b\}$$

The answer given is evidently incorrect as the determinant is of the n^{th} order and the power of $(b-a)$ should therefore be $n-1$.

Question 329.

(M. Bhimasena Rau) :—If the pedal circle of P with respect to a triangle ABC touches the nine point circle of ABC, show that the sum of the angles PAB, PBC, PCA, is constant.

Solution by N. Sankara Aiyar, M.A.

With the given condition the locus of P is given by

$$\Sigma x \cos A\,(y^2-z^2)=0,$$

(See p. 39, Vol. II, J.I.M.S.)

i.e., $\Sigma xy^2 \sin B \sin C=\Sigma(xy^2 \cos B \cos C+xz^2 \cos A).$

Now $\cos(A+B+C)=-1.$

$\therefore$ $\cos A \cos B \cos C+1=\Sigma \sin A \sin B \cos C.$

$\therefore$ $\Sigma(xy^2 \sin B \sin C)+xyz\Sigma(\sin A \sin B \cos C)$

$$=xyz(1+\cos A \cos B \cos C)+\Sigma(xy^2\cos B \cos C+xz^2 \cos A)$$

$\therefore$ $\Sigma xy \sin B \sin C\,(y+z \cos A)$

$$=(x+y\cos C)(y+z \cos A)(z+x \cos B).$$

$$\therefore \qquad \sum \frac{xy \sin B \sin C}{(x+y \cos C)(z+x \cos B)}=1.$$

If $PAB=\alpha$, $PBC=\beta$, $PCA=\gamma$, we get

$$\Sigma \tan \alpha \tan \beta=1.$$

$$\therefore \qquad \alpha+\beta+\gamma=90^\circ.$$

Hence the sum of the angles in question is constant and equal to a right angle.

Question 465.

(S. Narayana Aiyar, M.A.) :—When n lies between -1, and -2, shew that

$$(1)\ \int_0^\infty x^{n-1} \log \frac{\sinh x}{x}dx=\frac{\pi^{n+1}}{n \sin\frac{\pi n}{2}}\,S_n.$$

$$(2)\int_0^\infty x^{n-1} \log \frac{\cosh x-\cos x}{x^2}dx=\frac{2^{\frac{n}{2}}\ \pi^{n+1}S_n}{n \sin\frac{\pi n}{4}}$$

where $S_n=1^n+2^n+3^n$..................to infinity.

Solution by N. Sankara Aiyar, M.A.

Now
$$\frac{\sinh x}{x}=\prod_{r=1}^{\infty}\left((1+\frac{x^2}{r^2\pi^2}\right).$$

$$\therefore \qquad I=\int_0^\infty \Sigma x^{n-1} \log\left(1+\frac{x^2}{r^2\pi^2}\right)dx.$$

$$=\sum_{r=1}^{\infty}\left[\pi^n\, r^n\int_0^\infty y^{n-1} \log(1+y^2)\,dy.\right]$$

$$=\sum_{r=1}^{\infty}\left[\pi^n r^n\left\{\left[\frac{y^n \log(1+y^2)}{n}\right]_0^{\infty}-\frac{2}{n}\int_0^{\infty}\frac{y^n+1}{(1+y^2)}dy\right\}\right]$$

$$=\sum_{r=1}\left[-\pi^n r^n\frac{1}{n}\frac{2\pi}{2\cos\frac{(n+1)\pi}{2}}\right]$$

$$=\sum_{r=1}^{\infty}\frac{\pi^{n+1} r^n}{n\sin\frac{n\pi}{2}}$$

$$=\frac{\pi^{n+1}}{n\sin\frac{n\pi}{2}}S_n.$$

Again $\dfrac{\cosh x-\cos x}{x^2}=\prod_{r=1}^{\infty}\left(1+\dfrac{x^4}{4r^4\pi^4}\right).$

(Chrystal, Part II, p. 359.)

$$\therefore 1=\int_0^{\infty}\sum_{r=1}^{\infty}\left[x^{n-1}\log\left(1+\frac{x^4}{4r^4\pi^4}\right)dx\right]$$

$$=\sum_{r=1}^{r=\infty}\left[2^{\frac{n}{2}} r^n\pi^n\int_0^{\infty}y^{n-1}\log(1+y^4)\,dy\right]$$

$$=\sum_{r=1}^{r=\infty}\left[2^{\frac{n}{2}} r^n\pi^n\left\{\left[\frac{y^n}{n}\log(1+y^4)\right]_0^{\infty}-\frac{4}{n}\int_0^{\infty}\frac{y^{n+3}}{1+y^4}dy\right\}\right]$$

$$=\sum_{r=1}^{r=\infty}\left[2^{\frac{n}{2}} r^n\pi^n\frac{4\pi}{4n\sin\frac{n\pi}{4}}\right]=\frac{2^{\frac{n}{2}} n^{n+1}}{n\sin\frac{n\pi}{4}}S_n.$$

Question 510.

(S. P. Singaravelu Mudaliar):—Show that the locus of the symmedian point of the triangle of maximum area inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{4}\left(\frac{a^2-b^2}{a^2+b^2}\right).$$

Solution by T. P. Trivedi, M.A., LL.B., N. Sankara Aiyar, M.A., and D. Krishnamurti.

Let θ, $\theta+\frac{2\pi}{3}$, $\theta+\frac{4\pi}{3}$ be the vertices of the angular points A, B, C of the maximum triangle ABC in the ellipse.

We have

$$BC^2=a^2\left\{\cos\left(\theta+\frac{2\pi}{3}\right)-\cos\left(\theta+\frac{4\pi}{3}\right)\right\}^2$$
$$+b^2\left\{\sin\left(\theta+\frac{2\pi}{3}\right)-\sin\left(\theta+\frac{4\pi}{3}\right)\right\}^2$$
$$=3a^2\sin^2\theta+3b^2\cos^2\theta.$$

Similarly $\quad CA^2=3a^2\sin^2\left(\theta+\frac{2\pi}{3}\right)+3b^2\cos^2\left(\theta+\frac{2\pi}{3}\right),$

and $\quad AB^2=3a^2\sin^2\left(\theta+\frac{\pi}{3}\right)+3b^2\cos^2\left(\theta+\frac{\pi}{3}\right).$

The areal co-ordinates of the symmedian point are proportional to BC^2, CA^2, AB^2; thus if (x, y) be the symmedian point we have

$$\frac{x}{a}=\frac{BC^2\cos\theta.+CA^2\cos\left(\theta+\frac{2\pi}{3}\right)+AB^2\cos\left(\theta+\frac{4\pi}{3}\right)}{BC^2+CA^2+AB^2}$$

and $$\frac{y}{b}=\frac{BC^2\sin\theta.+CA^2\sin\left(\theta+\frac{2\pi}{3}\right)+AB^2\sin\left(\theta+\frac{4\pi}{3}\right).}{BC^2+CA^2+AB^2}$$

Now

$$BC^2+CA^2+AB^2=3a^2\left\{\sin^2\theta+\sin^2\left(\theta+\frac{\pi}{3}\right)+\sin^2\left(\theta+\frac{2\pi}{3}\right)\right\}$$
$$+3b^2\{\cos^2\theta+\ldots\ldots+\ldots\ldots\}$$
$$=\frac{9}{2}(a^2+b^2).$$

Again

$$\cos\theta\sin^2\theta+\cos\left(\theta+\frac{2\pi}{3}\right)\sin^2\left(\theta+\frac{2\pi}{3}\right)$$
$$+\cos\left(\theta+\frac{4\pi}{3}\right)\sin^2\left(\theta+\frac{\pi}{3}\right)$$
$$=\tfrac{1}{4}\left\{2\sin 2\theta\sin\theta+2\sin\left(2\theta+\frac{4\pi}{3}\right)\sin\left(\theta+\frac{2\pi}{3}\right)\right.$$
$$\left.-2\sin\left(2\theta+\frac{2\pi}{3}\right)\sin\left(\theta+\frac{\pi}{3}\right)\right\}$$

$$=\tfrac{1}{4}\left\{\cos\theta-\cos 3\theta+\cos\left(\theta+\frac{2\pi}{3}\right)-\cos(3\theta+2\pi)\right.$$
$$\left.-\cos\left(\theta+\frac{\pi}{3}\right)+\cos(3\theta+\pi)\right\}$$
$$=-\tfrac{3}{4}\cos 3\theta.$$

Similarly

$$\cos^3\theta+\cos^3\left(\theta+\frac{2\pi}{3}\right)+\cos\left(\theta+\frac{4\pi}{3}\right)\cos^2\left(\theta+\frac{\pi}{3}\right)$$
$$=\cos^3\theta+\cos^3\left(\theta+\frac{2\pi}{2}\right)-\cos^3\left(\theta+\frac{\pi}{3}\right)$$
$$=\tfrac{3}{4}\cos 3\theta, \text{ on simplification.}$$

Thus $$\frac{x}{a}=\frac{\frac{9}{4}\cos 3\theta(-a^2+b^2)}{\frac{9}{2}(a^2+b^2)}=-\frac{\cos 3\theta}{2}\frac{a^2-b^2}{a^2+b^2},$$

and $$\frac{y}{b}=-\frac{\sin 3\theta}{2}\frac{a^2-b^2}{a^2+b^2}.$$

Hence the locus is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{4}\left(\frac{a^2-b^2}{a^2+b^2}\right)^2.$$

Question 511.

(S. Krishnaswami Aiyangar) :—If S, H are the foci of the maximum inscribed ellipse of the triangle of reference, prove that

$$AS.BS.CS.AH.BH.CH=\frac{a^2b^2c^2}{27}.$$

Solution by N. Sankara Aiyar, M. A.

Now $AS.BS.CS.AH.BH.CH=16\,R^2\alpha^2\beta^2$.

(see p. 143, Vol. IV., J. I.M.S.)

But $$\alpha^2\beta^2=\frac{\Delta^2}{27}$$ (see p. 113, Vol. V.)

$$\therefore\quad AS.BS.CS.AH.BH.CH=\frac{16\,R^2\Delta^2}{27}=\frac{a^2b^2c^2}{27}.$$

Or thus

$$AS.AH=\frac{bc}{3}$$ (see p. 14, Vol. VI., J.I.M.S.)

$$\therefore\quad AS.BS.CS.AH.BH.CH=\frac{a^2b^2c^2}{27}.$$

Question 512.

(A. Narasinga Row) :—If $n\phi(n)+(n+2)\phi((n+2)=\phi(n+1)$, prove that

(i) $\phi(1)x-\phi(3)x^3+\phi(5)x^5\ldots\ldots=\phi(1)\sin(\tanh^{-1}x)$

(ii) $\phi(0)-\phi(2)x^2+\phi(4)x^4\ldots=\phi(0)+\phi(1)\ \{\cos(\tanh^{-1}x)-1\}$

Solution by N. Sankara Aiyar, M.A.

Let $y=\cos(\tanh^{-1}x)$

Differentiating $$y_1=-\frac{\sin(\tanh^{-1}x)}{1-x^2},$$

and $$y_2(1-x^2)-2xy_1=-\frac{\cos(\tanh^{-1}x)}{1-x^2}.$$

$\therefore$ $$y_2(1-x^2)^2-2xy_1(1-x^2)=-y.$$

i.e., $$y_2(1-2x^2+x^4)-2xy_1(1-x^2)+y=0.$$

Put $y=\Sigma A_r x^r$. Then, substituting and equating coefficients in the above equation, we get

$$(r+2)(r+1)A_{r+2}-(2r^2-1)A_r+(r-1)(r-2)A_{r-2}=0 \quad \ldots \quad (1)$$

Now it is clear that the differential equation is satisfied by $\sin(\tanh^{-1}x)=y$ also.

Taking $y=\sin(\tanh^{-1}x)$, we see that y changes sign with x and is zero when x is zero; hence it is an odd function of x.

Again differentiating $\sin(\tanh^{-1}x)=\Sigma A_{2r+1}x^{2r+1}$, we get

$$\cos(\tanh^{-1}x)=(1-x^2)\Sigma(2^{r+1})A_{2r+1}x^{2r}.$$

$\therefore$ $$A_{2r}=(2r+1)A_{2r+1}-(2r-1)A_{2r-1} \ldots \quad \ldots \quad (2)$$

Substituting in equation (1), we get

$$A_{2r+1}=(2r+2)A_{2r+2}-2rA_{2r},$$

making use of the similar condition got by differentiating the value of $\cos(\tanh^{-1}x)$ and equating to $\sin(\tanh^{-1}x)$.

Hence if $\sin(\tanh^{-1}x)=\Sigma(-1)A_{2r+1}x^{2r+1}$, the equation connecting successive A's is

$$A_r=(r+1)A_{r+1}+(r-1)A_{r=1};$$

or putting $\phi(n)$ for A_n

$$\phi(n+1)=n\phi(n)+(n+2)\phi(n+2).$$

Similarly $\cos(\tanh^{-1}x)=A_1+\Sigma(-1)^r A_{2r}$.

Hence we get

$$\phi(1)\sin(\tanh^{-}x)=\phi(1)x-\phi(3)x^3+\phi(5)x^5+\ldots\ldots$$

and $$\phi(1)\{\cos(\tanh^{-1}x)-1\}+\phi(o)=\phi(o)-\phi(2)x+\ldots\ldots$$

remembering that $A_1=1$.

Again put $x=iy$, then

$$\Sigma\phi(n)y^n=\phi(o)-\phi(1)+\phi(1)e^{-i\tanh^{-1}x}$$

$$=\phi(o)-\phi(1)+\phi(1)\,e^{-i\tanh^{-1}yi)}$$

$$=\phi(o)-\phi(1)+\phi(1)e^{\tan^{-1}y}.$$

In other words

$$\Sigma\phi(n)x^n=\phi(o)-\phi(1)+\phi(1)\,e^{\tan^{-1}x}.$$

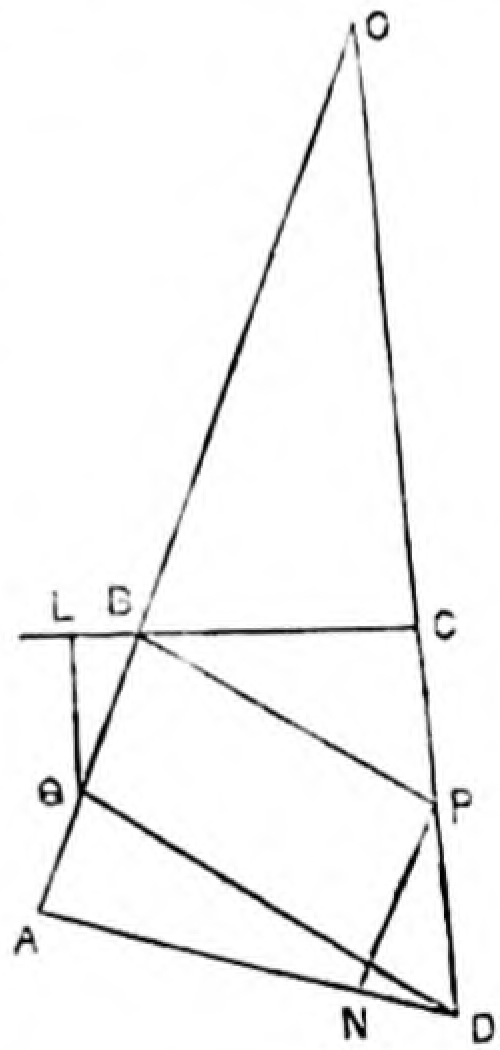
Hence the given series is always convergent.

Question 515.

(A. C. L. Wilkinson, M.A., F. R. A. S.):—ABCD is a quadrilateral. DQ, BP are any two parallel straight lines meeting AB, CD, respectively in Q and P. QL parallel to CDmeets BC in L, and PN parallel to AB meets AD in N. Prove that the middle points of AC, DL, BN are collinear.

Analytical solution by D. Krishnamurti.

Take OA, OD as the co-ordinate axes where O is the point of intersection of AB, CD.



Let $\left.\begin{array}{l} OB=a \\ OC=b \\ CP=k \\ PN=c \end{array}\right\}$ Then, if $BQ=\lambda\, a$, $PD=\lambda\,(b+k)$, since BP and QD are parallel.

Now the equation to DN is

$$\frac{x-c}{y-(b+k)}=-\frac{c}{\lambda(b+k)};$$

putting $y=o$, we get $OA=c+\dfrac{c}{\lambda}$.

Comparing Δ^s LBQ, CBO

$$QL=\lambda a.\frac{b}{a}=\lambda b.$$

Hence the co-ordinates of the following points are known

$B\,(a, o)$ $\quad C\,(o, b)$ $\quad L\,(a+a\lambda, -\lambda b)$

$N\,(c, b+k)$ $\quad A\left(c+\dfrac{c}{\lambda}, o.\right)$ $\quad D\,(o, \lambda+1\ b+k)$

The co-ordinates of the mid-points of BN, CA, LD are therefore

$$\left(\frac{a+c}{2}, \frac{b+k}{2}\right)\left(\frac{c\lambda+c}{2\lambda}, \frac{b}{2}\right)\left(\frac{a+a\lambda}{2}\ \ \frac{b+k+\lambda k}{2}\right)$$

If these are collinear then the usual determinant must vanish identically : multiplying by λ the column of the first determinant, we get

$$\Delta\equiv\begin{vmatrix} \lambda(a+c) & b+k & 1 \\ c(\lambda+1) & b & 1 \\ a\lambda(\lambda+1) & b+k+k\lambda & 1 \end{vmatrix}$$

Subtracting the 1^{st} row from each of the 2^{nd} and 3^{rd}, we get

$$\Delta \equiv -\lambda \begin{vmatrix} \lambda(a+c) & b+k & 1 \\ c-a\lambda & -k & 0 \\ (c-a\lambda) & -k & 0 \end{vmatrix} \equiv 0,$$

since two rows are identical.

Hence the mid-points of AC, DL, BN are collinear.

Question 516.

(K. J. Sanjana, M.A.) :—TP, TQ tangents to a conic of centre C and focus F, cut the auxiliary circle in Y, Z ; and FW is perpendicular to the chord of contact ; a tangent of the conic perpendicular to TC cuts FY, FZ, FW in y, z, w, respectively. Prove that $yw = wz$, and ennunciate the corresponding property for the parabola.

Additional solution by T. P. Trivedi, M.A., LL. B.

Let TC be joined. Since TC bisects PQ, the lines TP, TC, TQ and the parallel from T to PQ from a harmonic pencil.

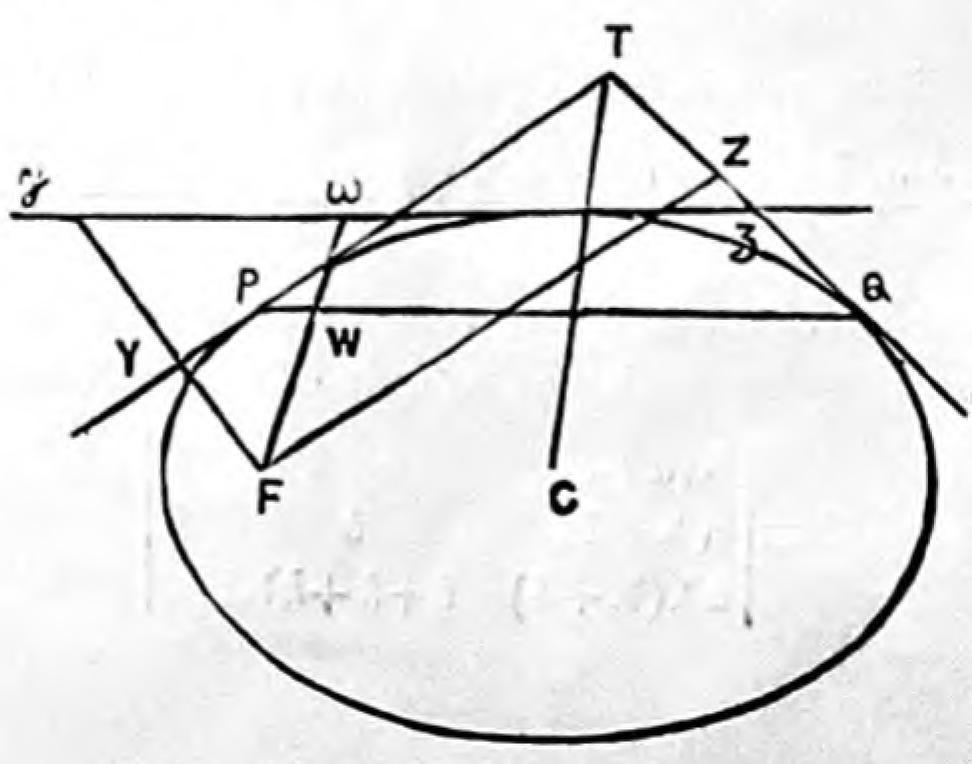
Join FY and FZ, which are perp. to TP and TQ respectively.

As FW is perpendicular to PQ, and as ywz is perp. to TC, a parallel from F to ywz will also be perp. to TC.

Thus FY, FW, FZ and a parallel from F to ywz from a harmonic pencil ; whence we infer $yw = wz$.

For the parabola this becomes :—

TP, TQ are two tangents to a parabola cutting the tangent at the vertex in Y and Z, and FW perpendicular to PQ meets the tangent at the vertex in W ; then YW = WZ.



QUESTIONS FOR SOLUTION.

562. (N. P. PANDYA) :—Solve the equation

$$2\left(5+9p^2+9y\frac{dp}{dx}\right)\frac{dp}{dx}+6(x+4py)\frac{d^2p}{dx^2}+(x^2+3y^2)\frac{d^3p}{dx^3}=\frac{18}{x^4}-\frac{24x(1-x^2)}{(1+x^2)^4},$$

where $p=\frac{dy}{dx}$.

563. (N. P. PANDYA) :—Given two quadrilaterals in the plane of the paper shew how to draw a straight line bisecting them both.

564. (E. H. NEVILLE, B.A., B.Sc.) :—If D denotes the determinant

$$\begin{vmatrix} 1 & m_1 & n_1 & a_1m_1 & a_1n_1 \\ 1 & m_2 & n_2 & a_2m_2 & a_2n_2 \\ 1 & m_3 & n_3 & a_3m_3 & a_3n_3 \\ 1 & m_4 & n_4 & a_4m_4 & a_4n_4 \\ 1 & m_5 & n_5 & a_5m_5 & a_5n_5 \end{vmatrix}$$

and D_r, M_r, N_r, P_r, Q_r are the co-factors of the elements of the r^{th} row, prove that

$$\begin{vmatrix} \Sigma(a_rm_rk_r), & \Sigma(m_rk_r) \\ \Sigma(a_rn_rk_r), & \Sigma(n_rk_r) \end{vmatrix}=\mathrm{D}\Sigma(k_r).$$

where k_r denotes $(M_rQ_r-N_rP_r)/D_r$ and the summations include the five values of r.

565. (S. KRISHNASWAMI AIYANGAR):—Shew that the locus of the orthopoles of tangents to the maximum inscribed ellipse of a triangle is a straight line passing through the orthocentre of the triangle.

566. (S. KRISHNASWAMY AIYANGAR):—Prove that

$$\sin^{-1}x=\frac{x}{(1-x^2)^{\frac{3}{2}}}\left\{1-s_1\frac{x^2}{1-x^2}+s_2\left(\frac{x^2}{1-x^2}\right)^2-\ldots\ldots\right\}$$

where s_r=the sum of the reciprocals of the first r odd numbers.

567. (P. A. SUBRAMANYA AIYAR, B.A., L.T.):—Shew that a uniform tetrahedron of mass m is equi-momental with six particles of mass $\frac{1}{10}m$ at the middle points of its six edges and one of mass $\frac{2}{5}m$ at the centre of mass of the tetrahedron.

568. (K. V. ANANTHANARAYANA SASTRI, B.A.):—Evaluate

$$\int_0^\infty \frac{\log x\, dx}{1+x^4}.$$

569. (R. VYTHYNATHASWAMY):—If p points are chosen at random on the circumference of a circle, find the chance of their lying on an arc equal to a k^{th} part of the circumference.

570. (Professor SANJANA) :—Prove that when n and k are positive integers the sums of the following series vanish—

$$(a)\ 1-\binom{n+2k}{2}\frac{1}{2n+2k-1}+\binom{n+2k}{4}\frac{1.3}{(2n+2k-1)(2n+2k-3)}$$

$$-\binom{n+2k}{6}\frac{1\cdot3\cdot5}{(2n+2k-1)(2n+2k-3)(2n+2k-5)}+\ldots\ldots\ldots,$$

$$(b)\ \binom{2n+2k+1}{1}-\binom{2n+2k+1}{3}\frac{2n+3}{2n+2k-1}$$

$$+\binom{2n+2k-1}{5}\frac{(2n+3)(2n+5)}{(2n+2k-1)(2n+2k-3)}-\ldots\ldots$$

where $\binom{x}{y}$ denotes ${}_xC_y$

571. (S. RAMANUJAN) :—If $\frac{\pi\alpha}{2}=\log\tan\left(\frac{\pi}{4}+\frac{\pi\beta}{4}\right)$, shew that

$$\left(\frac{1^2+\alpha^2}{1^2-\beta^2}\right)\left(\frac{3^2-\beta^2}{3^2+\alpha^2}\right)^3\left(\frac{5^2+\alpha^2}{5^2-\beta^2}\right)^5\ldots\ldots\ldots=e^{\frac{1}{2}\pi\alpha\beta}$$

572. (M. BHIMASENA RAO) :—If $f(x)=x+\frac{x^3}{3^2}+\frac{x^5}{5^2}+\frac{x^7}{7^2}+\ldots\ldots$, shew that

$$\text{(i)}\ f(\sqrt{2}-1)=\frac{\pi^2}{16}-\frac{1}{4}\{\log(\sqrt{2}-1)\}^2$$

$$\text{(ii)}\ f\left(\frac{\sqrt{5}-1}{2}\right)=\frac{\pi^2}{16}-\frac{3}{4}\left\{\log\frac{\sqrt{5}-1}{2}\right\}^2$$

$$\text{(iii)}\ f(\sqrt{5}-2)=\frac{\pi^2}{24}-\frac{2}{4}\left\{\log\frac{\sqrt{5}-1}{2}\right\}^2$$

573. (M. BHIMASENA RAO) :—Shew that

$$\frac{1}{2}-\frac{2}{3^2}\cdot\frac{1}{2^3}+\frac{2\cdot4}{3\cdot5^2}\cdot\frac{1}{2^5}-\frac{2\cdot4\cdot6}{3\cdot5\cdot7^2}\cdot\frac{1}{2^7}+\ldots\ldots=\frac{\pi^2}{12}-\frac{3}{2}\left\{\log\frac{\sqrt{5}-1}{2}\right\}^2.$$

574. (V. RAMASWAMI AIYAR, M.A.) :—Given a tetrahedron ABCD and a line LL′. Prove that the pedal spheres, with respect to the tatrahedron, of points lying in LL′ cut a fixed sphere, Σ, orthogonally. Defining the centre of the sphere Σ as the *orthopole* of LL′, and denoting it by σ, show that, in general, there are three points in LL′ such that the feet of the perpendiculars from any of them on the faces of ABCD are coplanar, and that the orthopole σ is the point of intersection of the pedal planes of these points. Show also that the locus of the orthopoles of a system of parallel lines (LL′) is a plane, Γ, at right angles to the system.

ERRATA TO VOL. V.

Page 3, line 8 from bottom, for '$t<0$,' *read* '$t>0$.'

Page 4, line 14, for comma after 'A,' have full stop.

Page 6, line 19, for the first ∞, *read* $-\infty$.

Page 11, footnote, for 'Chryslal' *read* 'Chrystal.'

Page 14, para 2 (1), for 'invesely' *read* 'inversely'; omit fullstop at the end of the same line.

Page 14, line 10 from bottom, for 'o' *read* 'O.'

Page 16, 3rd line from bottom, for 'exdansion' *read* 'expansion.'

Page 16, 2nd line from bottom, at the end, instead of the comma, *have* ....

Page 30, 6th line from bottom, for 'devisible' *read* 'divisible.'

Page 31, 1st line, for 'λ,' *read* 'y.'

Page 34, 2nd line from bottom, for '|' *read* '1.' Similarly on page 35, line 5.

Page 35, line 8 from bottom, for '0' *read* 'O.'

Page 35, 3rd line from bottom, for 'hypotheis' *read* 'hypothesis.'

Page 39, 3rd line from bottom, *add* 'c' at the end of the numerator.

Page 42, para 5, *insert* 'been' before 'presented.'

Page 50, 1st line of paragraph following second determinant, for '$AB,^2$ (,,' *read* '($AB,^2$.'

Page 52, line 18, for 'accounts' *read* 'accents.'

Page 66, Question 371, 1st line of solution for 'L, M, N. The' *read* 'L, M, N, the'

Page 67, line 4, for '$\underline{\psi}$' *read* 'T.'

Page 69, last line of solution to Question 401, for '8' *read* 'g.'

Page 69, last line, *insert* a bracket.

Page 71, line 8, for $\dfrac{\text{'}\pi^1\text{'}}{2}$ *read* $\dfrac{\text{'}\pi\text{'}}{2}$.

Page 83, 11th line, *read* '$\left(-\frac{3}{2}\right)$' in the index throughont.

Page 84, 14th line, for '*Quadric*' *read* '*Quartic*.'

Page 86, 2nd line, for the second '=,' *read* ':'

Page 103, line 4 from bottom, for 'question' *read* 'questions.'

Page 116, line 4, for 'controids' *read* 'centroids.'

Page 113, line 4, *omit* the comma after '*Durai*.'

Page 132, 6th line from bottom, for $\dfrac{\text{' '}}{2}$ *read* $\dfrac{\text{'}3\text{'}}{2}$.

Page 133, line 8, for '*farmula*' *read* '*formula*.'

Page 133, line 7 from bottom, for 'Briggt' *read* 'Briggs.'

Page 133, line 6 from bottom, for 'facs,' *read* 'fact.'

Page 133, line 3 from bottom, for 'cord' *read* 'chord.'

Page 139, line 12, for 'quadradic' *read* 'quadratic.'

Page 140, line 14, for 'z^2' *read* 'z^4.'

Page 156, under Question 446, for 'Claraiut' *read* 'Clairaut.'

Page 159, line 4 from bottom, for the comma at the end, *read* 'l.'

Page 161, 2 (c) 2, for 'Mechanices' *read* 'Mechanics.'

Page 164, line 9 from bottom, for 'mathmatician,' *read* 'mathematician.'

Page 164, line 5 from bottom, for 'distance' *read* 'distances.'

Page 164, line 2 from bottom, for 'fülls' *read* 'fills.'

Page 164, footnote, 1st line, for 'earliar' *read* 'earlier.'

Page 165, line 18, for 'pooe,' *read* 'poor.'

Page 166, line 13 from bottom, insert a comma at the end.

Page 166, line 3 from bottom, for 'wether' *read* 'whether.'

Page 167, *omit* the 'of' at the end of line 1.

Page 167, line 7, for 'meager' *read* 'meagre.'

Page 167, line 12 from bottom, for 'mathematial' *read* 'mathematical.'

Page 167, line 7 from bottom, for 'reson' *read* 'reason.'

Page 169, line 20, for 'principal' *read* 'principle.'

Page 170, line 5 from bottom, for 'to' *read* 'two.'

Page 170, line 5 from bottom, for 'extention' *read* 'extension.'

Page 170, last line for 'Ssciety' *read* 'Society.'

Page 171, line 8, for 'three,' *read* 'there.'

Page 174, line 6, for 'publications' *read* 'publication.'

Page 177, line following the results (4), *insert* 'and' between the 'r's' and 'the 'λ's.'

Page 178, line 11, for '$\frac{e^{3x}}{|2}$' *read* '$\frac{e^{2x}}{\underline{|2}}$'.

Page 194, line 10, for 'as' *read* 'are.'

Page 203, Ex. 2, 3rd line for (1, 1, 00), *read* (1, 1, 0, 0).

Page 206, line 1, for 'Lagranges' *read* 'Lagrange's.'

Page 213, line 7, for 'cone,' *read* 'done.'

Page 227, line 9, *omit* '±.'

Page 233, line 2, for 'spherica.,' *read* 'spherical.'

Page 238, 2nd line of solution, for 'long' *read* 'log.'

List of Periodicals Received.

(From 16th May to 15th July 1914.)

1. Acta Mathematica, Vol. 37, No. 3.
2. American Journal of Mathematics, Vol. 36, No. 2, April 1914.
3. Annals of Mathematics, Vol. 15, No. 4, June 1914.
4. Astrophysical Journal, Vol. 39, Nos. 3 and 4, April and May 1914.
5. Bulletin of the American Mathematical Society, Vol. 20, Nos. 8 and 9, May and June 1914.
6. Bulletin des Sciences Mathematiques, Vol. 35, May and June 1914.
7. Educational Times, June 1914, (6 copies).
8. L'Education Mathematique, Vol. 16, Nos. 14, 15, 16 and 17.
9. Fortschritte der der Mathematik, Vol. 42, No. 3.
10. L'Intermediaire des Mathematicians, Vol. 21, Nos. 4 and 5, April and May 1914.
11. Journal de Mathematiques Elementaires, Vol. 38, Nos. 14, 15, 16 and 17.
12. Mathematical Gazette, Vol. 7, No. 111, May 1914, (4 copies.)
13. Mathematics Teacher, Vol. 6, No. 3, March 1914.
14. Mathematische Annalen, Vol. 75, No. 3, June 1914.
15. Mathesis, Vol. 4, April and May 1914.
16. Messenger of Mathematics, Vol. 43, Nos. 10, 11 and 12, February, March and April 1914.
17. Monthly Notices of the Royal, Astronomical Society, Vol. 74, Nos. 5, 6, and 7, March, April and May 1914.
18. Philosophical Magazine, Vol. 27, Nos. 161 and 162, May and June 1914.
19. Popular Astronomy, Vols 22, Nos. 5 and 6, May and June—July 1914, (3 copies).
20. Proceedings of the London Mathematical Society, Vol. 13, Nos. 3 and 4, April and June 1914.
21. Proceedings of the Royal Society of London, Vol. 90, Nos. 616, 617 and 618, April and May 1914.
22. Quarterly Journal of Mathematics, Vol. 45, No. 3, May 1914.
23. Revue de Mathematiques Speciales, Vol. 24, Nos. 8 and 9, May and June 1914.
24. School Science and Mathematics, Vol. 14, Nos. 5 and 6, May and June 1914, (3 copies).
25. Transactions of the American Mathematical Sosciety, Vol. 15, Nos. 1 to 2, January and April 1914.
26. Transactions of the Royal Society of London, Vol. 214, Nos. 511, 512 and 513.
27. The Tohoku Mathematical Journal, Vol. 5, Nos. 1 and 2, May 1914.
28. Nature, Vol. 93, Nos. 2320-26 and 2328 and 2329, April, May and June 1914.
29. American Mathematical Monthly, Vol. 21, Nos. 1 to 5, January to May 1914.

The Indian Mathematical Society.

(Founded in 1907 for the Advancement of Mathematical Study, and Research in India.)

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Hony. Joint Secretary,

WITH THE CO-OPERATION OF

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CONTENTS.

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Vol. VI.] OCTOBER 1914. **[No. 5.**

PROGRESS REPORT.

Mr. V. G. Dalvi, B.A., (Bombay and Cantab)—Chief Secretary to the Indore Durbar, Indore,—has been elected a member of our Society.

2. As some numbers of the "American Mathematical Monthly" for 1913 were received last year, it was considered desirable to order the remaining numbers, and now the whole volume for 1913 is available for issue at the Library to members desiring to have it.

3. "*Matriculation Mechanics*"—by Briggs and Bryan, University Tutorial Press, 3rd edition, 1914, 3*s.* 6*d.*—is presented to our Library by the Publishers.

Calcutta University Calendar, Part III, 1914—has likewise been presented to the Library.

POONA,
30th September 1914.

D. D. KAPADIA,
Hony. Joint Secretary.

The Theories of Irrational Numbers, Part I:

Incommensurables, Convergent, and Earlier Geometrical Theories.

By Phillip E. B. Jourdain.

The aim of this historical and critical study is somewhat different from that of the other works that are known to us which deal with the development of the theory of convergence and allied topics. We shall, in fact, be concerned primarily with questions of principle; and the numerous and valuable deductions as to the criteria which enable us to decide whether such and such a series or class of series is convergent or not, which belong to technical mathematics, will only be shortly referred to.

The whole theory of convergence rests on one principle which is generally referred to as "Cauchy's general criterion," although the name of Bolzano might, perhaps, with greater propriety, be associated with this criterion. On the basis of this general criterion, various special criteria have been set up by Cauchy and many others for the purpose of deciding whether certain series do in fact converge, and a general theory of these special criteria has been the work of Dini, Paul du Bois-Reymond, and Pringsheim. But in all this work the validity of the general criterion itself was presupposed. This validity seemed, indeed, quite obvious to "intuition," and was either examined insufficiently or not at all, until there arose mathematicians who felt the need of tracing the logical connexion of mathematical theorems to the fundamental laws of arithmetic or even logic itself. Then it became evident that we cannot accept without a strict logical proof the theorem that, if a series fulfils such and such a condition, that series has a limit. From a logical point of view, the proposition that there is a limit is not in the very least proved by an appeal to those geometrical notions which seem to make it extremely plausible that the well-known geometrical interpretation of our conception of an arithmetical series points to something for which the arithmetical correspondent—if there be one—would be the limit in question. We must actually construct the entity we require out of our conceptions, and actually produce it for the mental vision of others. An appeal to "geometrical evidence," quite apart from its irrelevance, resolves itself into an appeal to faith, and the appeal is all the more vicious because, owing to certain psychological circumstances, it is almost universally responded to without reflection.

This first part is devoted, in the first place, to a sketch of the discovery of incommensurable magnitudes, the consequent severance, by the logically minded Greeks of ancient times, of geometry from arithmetic, and the gradual assumption of the exact correspondence between arithmetic and geometry, owing principally to the discovery of analytical geometry (§ I). In the second place, the rise of the careful treatment of the convergence of series is described. After a reference (§ II) to the work of Waring, Lagrange, and Gauss, which has been partially treated in other works on theh istory of mathematics, we proceed to a more detailed account of the contributions of Fourier (§ III), Cauchy (§ IV), and Bolzano (§ V) to the theory of the convergence of series. With Bolzano we have a very important indication of the fact that the existence of a limit to a series which fulfils a certain well-known condition must be proved, and the subsequent attempts of Hankel and Stolz to prove it are then dealt with (§ VI). Stolz's "proof" was founded on the conception of the least interval that includes all the limits to which a series that converges towards many limits can converge. This conception, which was known to Cauchy and Abel, had been forgotten, and was re-introduced by Paul du Bois-Reymond (§ VII). Du Bois-Reymond and Stolz apparently did not realize that the question of principle was the same as in the less complicated case of one limit, and it is instructive to study their errors and obscurities on this point. After a characterization (§ VIII) of the main purpose of theories of irrational numbers, the introduction of irrational numbers in De Morgan's and other text-books is examined (§ IX), and the geometrical theories (Guilmin, Bertrand) published before 1872 described (§ X).

§ I.

Pythagoras and the Pythagoreans held that number is the essence of all things, and that there is a complete parallelism between the conception of number and geometrical representation, in the sense that it is always possible to express the lengths of two straight lines as multiples of the same unit, provided that this unit was chosen small enough. However, Pythagoras himself discovered[1] and his discovery is one of the greatest of antiquity—that the diagonal of a square is incommensurable with its side; that is to say, no two numbers can express the ratio of these lines, so that the diagonal cannot be represented as a sum of points. The Pythagoreans seem to have believed that all geometrical entities could be so represented. This discovery was probably made in consequence of the discovery by Pythagoras of the relation between the

[1] Moritz Canto, *Vorlesungen über Geschichte der Mathematik*, Vol. I, 3rd, ed. Leipzig, 1907, pp. 181-184.

square on the hypotenuse and the squares on the other sides of a right-angled triangle, a theorem of which a particular case seems to have been known to the Babylonians[2] Chinese, and Egyptians, and which seems to have been arrived at in its generality, though probably merely by induction, by the Hindus.[2] If the sides of the triangle which include the right-angle between them are both of unit length, so that the hypotenuse is a diagonal of the unit square, it is obvious that the numerical measure of the length of this diagonal would be such that its square would be equal to 2. All attempts to find a fraction which should be the square root of 2 failed, and it would seem that at last the following proof of the incommensurability of the diagonal and the side was found.[3] Suppose that the diagonal were to the side as the integer d to the integer c, the ratio d/c being in its lowest terms. The Pythagorean theorem gives us the equation $d^2=2c^2$, so that we must conclude, that d is even and hence that c is odd. But, since d is even, we can write the theorem of Pythagoras as:

$$4(d/2)^2=2c^2 \text{ or } 2(d/2)^2=c^2,$$

which shows that c should be even. Hence the hypothesis that the ratio d/c expresses, in its lowest terms, the ratio of diagonal to side, must be given up.

At this point came the arguments of Zeno the Eleatic, which demonstrated to the Greeks the impossibility of making discontinuous plurality—the Pythagorean plurality of arithmetical points—account for the given continuous reality of space.[4] It would seem that Zeno—a supporter of the doctrine held by his master Parmenides that plurality is to be denied and that reality is unchanging—sought to show that motion is impossible. These arguments appear to have played a part in the necessity which the logically-minded Greeks felt for a rigid separation of arithmetic from geometry. These two sciences were only recombined after about eighteen hundred years of logical stagnation had dulled logical perception. But, perhaps, it was not this so much as it was a characteristic of the later and more western civilizations to pay less attention to logic while paying more to the observation of natural facts like motion.

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[2] *Cf.* Leon Brunschvicg, *Les étapes de la philosophie mathématique* Paris, 1912, p. 46.

[3] For the reasons for supposing that this proof (Euclid, *Elements*, Book X Prop. 117) dated from Pythagorean times, see Cantor, *op. cit.*, pp. 182-183 Brunschvicg, *op. cit.*, p. 47.

[4] Brunschvicg, *op. cit.*, p. 48; cf. pp. 154-155.

The Pythagorean discovery seems to have led, in spite of the Pythagoreans, to the rejection of the Pythagorean doctrine that numbers are fundamental to all reality, and to the adoption, by Plato, of a system in which geometry was fundamental.[5] Euclid[6] expressly said that incommensurable magnitudes are not related to one another in the same way as numbers. If, indeed, we represent the rational numbers such that their nth powers approximate to a given fraction which has no rational nth root, by points on a straight line, the sequence of points clusters round a point which does not correspond to any rational number.

The arithmeticians and algebraists of the middle ages and the Renascence—such as Leonardo of Pisa in his *Liber Abaci* of 1202—spoke of irrationals as "fictitious numbers" or "absurd numbers."[7] We find irrational "numbers" spoken of by Michæl Stifel in his *Arithmetica Integra* of 1544. He said that "an irrational number is not a true number," but this merely means that it is not a rational number. Stifel also remarked that every irrational number, just as every rational number, has a uniquely determined place in the series of numbers.

This is characteristic of the difference in the point of view of more modern mathematicians from that of the ancient Greeks. A tendency is observable to sacrifice the strict logic of the Greeks for the end of extending conceptions and methods into domains where it was felt—and perhaps only vaguely felt—that they applied. Thus the conception of a "variable" in mathematics was first introduced by the endeavour to include motion in space among the objects dealt with by mathematics. We know that the Greeks had often used the conception of motion in geometry as a means of discovery, but it was first in the work of Galileo that the conception in question began to be systematically used.

The discovery of analytical geometry by Fermat and Descartes and the discovery of the infinitesimal calculus by Leibnitz and Newton made it necessary to assume that every line could be measured in length by a number and thus practically identified geometry with the algebra of that time.[8] By the uncritical assumption of this correspondence bet-

[5] *Ibid.*, pp. 45, 47, 48.

[6] *Elements*, Book X, Prop. 7. The fifth and tenth Books of the *Elements* are concerned, respectively, with the general theory of ratios and incommensurable magnitudes.

[7] From the latter name—"numeri surdi"—the English name of "surd" is derived.

[8] See A. Pringsheim *Encykl. der. math. Wiss.*, I.A. 3, pp. 49-53; and in greater detail, in the French edition (Pringsheim and Molk), I. 3, pp. 133-147.

ween numbers and lengths of lines, the distinction so clearly seen by the Greeks was slurred over and forgotton.

Newton, in his *Arithmetica Universalis* of 1707, understood by "number" a ratio of two quantities; and, up to within quite recent times, the introduction of irrational "number" was, explicitly or implicity geometrical.[9] We shall see later that Cauchy held the same opinion.

The conception of a "limit"[10] grew out of the "principle of exhaustion" used by the ancient Greeks, and the first to formulate the conception arithmetically was John Wallis. In his *Arithmetica Infinitorum* of 1655, he defined a number a to be a "limit" of a infinite sequence a_0, a_1, a_2,..., if the difference $a-a_n$ becomes arbitrarily small when the integer n has a sufficiently great value. This definition, which is practically the same as that used at the present time, evidently presupposes the existence of a; but Cauchy was apparently the first to indicate the necessity of proving in an arithmetical way the existence of a limit. This he did in his proof (1823) of the existence of a definite integral of a continuous function. However, with Cauchy the question is only reduced to the assumption of the existence of a limit when a certain simpler criterion is fulfilled; and it is with this assumption that we are concerned in the present question of principle.

§ II.

The theory of convergence of series is principally due to mathematicians of the nineteenth century. Before this, only a few scattered theorems, due for the most part to Waring and contained in his *Meditationes analyticae* of 1781,[11] were known; and even Lagrange thought, in 1770,[12] that it is sufficient for the convergence of a series that its terms approach zero indefinitely. The introduction, by Lagrange in 1797, of a formula expressing the remainder after n terms of Taylor's series was not with a view to proving its convergence—for, this was a point upon which Lagrange did not touch—, but in order to be able to estimate the limit of error when we break off with a finite number of terms.

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[9] See *Ibid.*, pp. 153-154.

[10] *Cf.* Pringsheim, *Encykl. der. math. Wiss.*, I.A. 3, pp. 63-65.

[11] *Cf.* M. Cantor, *Vorlesungen uber Geschichte der Mathematik*, Vol. IV., Leipzig, 1908, pp. 275-276, 285-287 (article by E. Netto). On the history of convergence, see, besides, the third and fourth volumes of Cantor's *Geschichte*, R. Reiff, *Geschichte der unendlichen Reihen*, Tubingen, 1889; A. Pringsheim, *Encykl. der math. Wiss.*, I A. 3, 1898, pp. 78-79, and the corresponding part of the French edition. Reiff did not mention Waring.

[12] Reiff, *op. cit.*, p. 148.

Still, Lagrange's *Theorie des Fonctions* of 1797 may be said [13] to form the passage from the old to the modern treatment of infinite series, and Lagrange's investigations on the remainder of Taylor's series appear to have influenced Fourier and, partly through him, Cauchy, to evolve a strict theory of convergence.

Gauss, in his inaugural dissertation of 1799, said, in his criticism of a procedure of d'Alembert's for finding the form of the roots of an algebraic equation:[14] "Infinitely small magnitudes were used by d'Alembert in a looser way than is compatible with mathematical rigour, or than a far-seeing mathematician would allow at the present time, when those magnitudes are rightly regarded with mistrust. Also the leap from an infinitely small to a finite value of Ω [an infinitely small value of the rational and whole function considered] is not clear enough. The assertion that Ω can receive any finite value cannot be concluded from the possibility of an infinitely small value of Ω. This assertion follows from the fact that, when Ω is small enough, the approximation to the true value of ω [the value of x corresponding to Ω] increases with the number of the terms [of the series for ω in terms of Ω) retained, because of the strong convergence of the series. In other words, the equation which gives the relation between ω and Ω (or between x and X) is satisfied so much the more nearly as we take more terms to get ω. But this whole method of drawing conclusions seems too indefinite for any rigorous inference to be drawn from it. Besides this, I may also remark that there are series which always diverge however small the value of the magnitude in powers of which the development is effected, so that if we proceed far enough, we arrive at terms which are greater than any given magnitude. This happens if the coefficients of the series form a hypergeometric series."

Gauss added a note remarking, by the way, that there are series "which first converge very strongly and then more and more weakly, and finally diverge more and more; but, in spite of this, give the sum almost exactly if not too many terms are taken." He mentioned that this fact did not seem to have been noticed before, and that it was desirable to show the reason of this behaviour and to determine in how far such a sum might be securely taken as correct. Again, Gauss remarked[15] that "from the supposition that X can take the value S but not the value U does not follow that between S and U a value T must lie which X can reach but not surpass. Here a case is overlooked: it is

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[13] *Ibid*, pp. 153, 159.

[14] *Ostwald's Klassiker*, No. 14, p. 11.

[15] *Ibid.*, p. 12.

possible that between S and U there lies a limit to which U can approach arbitrarily near without reaching. From the grounds advanced by d'Alembert only follow that X can surpass every value which it reaches by a finite magnitude. If therefore it is, say S, it can be increased by a finite magnititude Ω: then we can have a new increase Ω', and then a further increase Ω'' and so on. But there is no last increase. However many increments have been added, a new one can always be added. But, though the number of the possible increments is infinite, it may happen that, if $\Omega', \Omega'', \ldots$ continually decrease, the sum

$$S+\Omega+\Omega'+\Omega''+\ldots$$

never reaches a certain limit, however many terms may be taken. This case cannot indeed occur if X is a whole algebraic function of x, but it must be proved that the case in question cannot happen...".

We shall refer to this passage again when we come to consider Bolzano's formulation of the conception of "upper limit." At present, it merely serves the purpose of showing that the difficulties which were brought forward in the first two paradoxes of Zeno about motion, and which are only solved by the ability to conceive an infinite series converging to a finite limit, were most decidedly overcome by Gauss.

Gauss, in his memoir on the hypergeometric series, published in 1812, used the principle of comparison of a series with the geometric series, which is known to have a finite and assigned limit under a certain simple condition[16]. This principle requires for its rigorous foundation one of the modern theories of irrational number. Of course, we cannot think it probable that Gauss had penetrated so deeply into the foundations of analysis as to have evolved a sound theory of irrational numbers, but it is certain that he had formed the idea that it is necessary to assure oneself that the sum of the terms of a series approach a finite limit before we can calculate with this limit—"the sum."

§ III.

In 1811, Fourier[17] gave a definition of convergence which showed that he had correct and fairly precise ideas on the conditions under which "the sum" of an infinite series could be calculated with Fourier's work was not actually printed until 1822, and it is impossible to say how far, if at all, Cauchy's conception of convergence, which was published in 1821, was indebted to that of Fourier.

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[16] Reiff, *op. cit.*, p. 162. Gauss (see *ibid.*, pp. 163-166) actually continued the investigation of convergence so far as to decide the convergence or otherwise of the series considered by him, in every case.

[17] Fourier presented a memoir on the theory of the conduction of heat to the French Institute in 1807 (on it cf. below and *Oeuvres de Fourier*, Vol. I, p. iv;

Speaking of a certain trigonometrical series in his *Theorie analytique de la Chaleur* of 1822, Fourier said: "It would be easy to prove that this series is always convergent, that is to say that writing, instead of y, any number whatever, and following the calculation of the coefficients, we approach more and more to a fixed value; so that the difference of this value from the sum of the calculated terms becomes less than any assignable magnitude. Without stopping for a proof,

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Vol. II, pp. vii, xii, 215-221). For 1811, the French Academy proposed for a prize the finding of a mathematical theory of the propagation of heat, and a long essay by Fourier, which was deposited in the archives on Sept. 28th 1811, was awarded the prize in 1812 (see. *Ibid*, Vol I, pp. vii-viii). When, after Delambre's death, Fourier became perpetual secretary, he had the memoir printed without any alteration, in the *Memoires de l'Académie Royale des Sciences de l'Institute d France* (vol. IV, 1819 and 1820, published 1824. pp. 185-556, and Vol. V, 1821 and 1822, published 1826, pp, 153-246), after the publication (1822) of his *Theorie analytique de la Chaleur* (we shall also refer to the English translated by A. Freeman cited below). Fourier doubtless wished to establish thus, in an incontestible manner, his rights of priority, for the first part of the memoir of 1811 (that which appeared in the volume for 1819-20) only differs in quite secondary points from the definitive version in the *Theorie de la Chaleur*. Thus Darboux did not reproduce the first part; but, as the second, printed in the *Memoires* for 1821-2, is of greatest interest, it is reproduced in the *Oeuvres de Fourier*, Vol. II, pp. 1-94.

Here we have to consider Fourier's precise conception (cf. Pringsheim, *Encykl. der math. Wiss.*, I. A. 3, p. 146, where the reference is wrong) of convergence, on p. 315, pages 313-315 are exactly reproduced in article 228 of the *Théorie* (*Oeuvres*, Vol. I, pp. 221-222; *Freeman* pp. 195-197,—see below). A careful comparison of Fourier's memoir of 1811 with the *Théorie* has also shown:

(1) That the word "convergent is used on p. 269 of the *Mémoires* (which corresponds to article 177 of the *Théorie*), but the explanation of it, which is given in the *Théorie*, is not given in the *Mémoires*;

(2) Article 178 of the *Théorie* is exactly given on p. 270 of the *Mémoires*;

(3) Articles 184-186 of the *Théorie* are given, in essentials, on pp. 275-278 of the *Mémoires*;

(4) Divergent series are used for calculation on, for example, p. 262 of the *Mémoires*, which exactly corresponds to article 171 of the *Théorie* (*Oeuvres*, Vol. I., p. 149);

(5) An oscillating series is used for calculation (in fact, for $1-1+1-\ldots$ is put $\frac{1}{2}$) in the *Memoires*, as in article 218 of *Theorie* (see *Oeuvres*, Vol. I, p. 206 *Freeman*, p. 182). But the calculation with an oscillating series in article 235 of the *Theorie* (see *Oeuvres*, Vol. I, p. 234; *Freeman* p. 288) is not in the *Memoires*;

(6) The Geometrical construction given by Fourier for proving the convergence of his integrals and series (cf. the text below) were intentionally omitted from the memoir of 1811. Fourier's modern conception of convergence certainly dated from before 1811, but no statement dating from an earlier year has been hitherto published.

which the reader may supply..."[18]. Again, convergence, says Fourier.[19] 'does not result solely from the fact that the values of the terms diminish continually; for, this condition is not sufficient to establish the convergence of a series. It is necessary that the values at which we arrive on increasing continually the number of terms should approach more and more a fixed limit, and should differ from it only by a quantity which becomes less than any given magnitude: this limit is the value of the series."

In article 178 of the *Theorie*,[20] Fourier gave an indication of the way in which the curve represented by a finite number of terms of the series

$$\cos x - \tfrac{1}{3}\cos 3x + \tfrac{1}{5}\cos 5x - \ldots$$

behaves as this number is increased; and, in the immediately following third section of the third chapter,[21] Fourier summed the first m terms of the above series, regarded this sum as a function (y) of x and m, and by a differentiation, certain other operations, integration by parts, expressed y as an infinite series arranged according to negative powers of m. Then it appears that, the more m increases, the more y approaches a constant value independent of x. Darboux[22] remarked on this passage that, in the study of some particular series, Fourier followed exactly the same method, which Dirichlet later used, to provide, for the first time, a rigorous theory of trigonometrical series.[23]

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[18] *Theorie*, art. 177; *Oeuvres de Fourier*, Vol. I, p. 156; English translation, with notes by R.L. Ellis and Freeman, by Alexander Freeman (*The Analytical Theory of Heat*, Cambridge 1878; referred to here as *Freeman*), pp. 143-144. A passage on p. 433 (art. 418; *Oeuvres*, Vol. I. p. 503) is to the same effect; while a passage on p. 207 (art. 235; *Oeuvrec*, Vol. I, p. 282), which runs: "The series arranged according to sines or cosines of multiple arcs are always convergent; that is to say, on giving to the variable any value whatever that is not imaginary, the sum of the terms converges more and more to a single fixed limit, which is the value of the developed function," is of interest as showing that the series which Fourier considered only converge along the real axis (cf. P. du Bois-Reymond's paper of 1876 reprinted in No. 186 of *Ostwald's Klassiker*, pp. 6-8).

[19] *Theorie*, art. 228; *Oeuvres*, Vol. I, pp. 221-22; *Freeman*, pp. 196-197.

[20] *Oeuvres*, Vol. I, pp. 157—158; *Freeman*, p. 144. In the *Memoires*, this passage occurs on pp. 269—270.

[21] "Remarques sur ces séries", *Oeuvres*, Vol. I, pp, 158—169; *Freeman*, pp. 145—154. These investigations are on pp. 270—280 of the *Memoires*.

[22] *Oeuvres de Fourier*, Vol. I, p. 158 *n*.

[23] This method is to express by a definite integral the sum of the m first terms of the series and then to seek the limit of this integral.

Fourier, after treating other analogous series in a like manner, proceeded[24] to determine the limits between which the remainder of such a trigonometrical series lies.

In his first investigations, which were presented to the Institute of France in 1807, or in the extensive notes, which were partly concerned with the convergence of series and which Fourier added in 1808 or 1809 to the original memoir[25] to reply to or forestall the objections of Lagrange and Laplace,[26] Fourier gave some geometrical constructions for demonstrating the convergence of his double integrals and his series, which were left out of the memoir of 1811.[27]

That this geometrical argument of Fourier's[28], although it is not carried through rigorously, contains the germs of Dirichlet's celebrated investigation, was remarked by Reiff[29] and Darboux.[30]

§ IV.

Before the time of Weierstrass, the introduction of irrational numbers was, explicitly or implicitly, geometrical. We will first examine Cauchy's *Cours d'Analyse* of 1821.[31]

In the "Préliminaires", Cauchy explicitly said[32] that the word "number" was to be understood to denote the entities which arise from the measurement of magnitudes. Again, he said[33] "Just as the idea of number arises from the measurement of magnitudes, so that of *quantity* arises when we consider each magnitude of a given species as serving for the increment or diminution of another fixed magnitude of the same species. We will agree to range absolute numbers, which are not preceded by any sign [denoted by capital letters], in the class of positive quantities." Relative numbers (numbers with sign or "quanti-

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[24] *Théorie*, arts. 185-188; *Oeuvres*, Vol. I, pp. 165-168; *Freeman*, pp. 150-153. This investigation is given in the *Memoires* pp. 275-280.

[25] *Théorie*, "Discours préliminaire"; *Oeuvres*, Vol. I, p. xxvi; *Freeman*, p. 9.

[26] *Oeuvaes de Fourier*, Vol. II, p. vii.

[27] *Thorie*, "Discours" and art. 415; *Oeuvres*, Vol. I, pp. xxvi., 404; *Freeman*, pp. 10, 426.

[28] *Theorie*, art. 423; *Freeman*, pp. 438-440; *Oeuvres*, Vol. I, pp. 509-513.

[29] R. Reiff, *op. cit.*, pp. 184-186.

[30] *Oeuvres de Fourier*, Vol. I, pp. 499, 511-514; cf. the note on p. 158 referred to above.

[31] *Cours d'Analyse de l'   le royale Polytechnique*: Première Partie [no further parts appeared], *Analyse algebrique*, Paris, 1821, Chap. VI, *Oeuvres* (2), Vol. III, Paris, 1897.

[32] *Ibid*, p. 17.

[33] *Ibid*, p. 333.

ties ") were denoted by small italicized letters; and Cauchy[54] remarked that we must carefully distinguish between operations relating to numbers and those relating to quantities.

Near the beginning of the book, Cauchy defined a "limit" as follows: "When the successive values attributed to a variable approach indefinitely a fixed value, so as to end by differing from it as little as wished, this last is called the 'limit' of all the others"; and remarked that "thus an irrational number is the limit of the various fractions which furnish more and more approximate values of it." If we consider—as, however, Cauchy did not do, although many others have done—the latter statement as a definition, so that an 'irrational' number is defined to be the limit of certain sums of rational numbers, we presuppose that this sum *has* a limit. In a note at the end of the book, Cauchy[55] explained the multiplication of the number A by the number B, where B is irrational: "...We can obtain, in terms of rational numbers, values which are more and more approximate to B. We easily show that, under the same hypothesis, the product of A by the rational numbers in question approximates more and more to a certain limit. This limit will be the product of A by B." And the explanation of A^B, when B is an irrational number, is[56] on the same principles.

Thus, Cauchy did not define real numbers arithmetically, but presupposed their existence, by assuming a correspondence between the system of real numbers and the system of points on a straight line.[57]

Towards the beginning of the fifth chapter of Cauchy's work, he[58] got, for instance, from the functional equation between continuous functions

$$\phi(x+y)=\phi(x)+\phi(y),$$

the equation $\phi(x)=x\phi(1)$, by proving the latter for rational x's, then varying x so as to approach an irrational limit, and then "passing to the limit." The legitimacy of this passage depends on the theorem:[59] If $x, y,$...converge respectively towards the limits $X, Y,$... ; and $f(x, y,...)$ is continuous with respect to each variable in the neighbourhood of the system of values $x=X$, $y=Y$,...$f(x, y,...)$ will have $f(X, Y,...)$ as limit. Cauchy's method of conclusion by "passing to the limit" may also be illustrated by his argument, at the end of § 1 of the sixth chapter of

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[54] *Ibid*, p. 335.

[55] *Ibid.*, p. 337.

[56] *Ibid.*, pp. 341-342.

[57] For a critique of this note of Cauchy's, see H. Hankel, *Theorie der complexen Zahlensysteme*, Leipzig, 1867, p. 14.

[58] *Op. cit.*, p. 99; cf. pp. 102, 108-109, 111.

[59] *Ibid.*, p. 47.

his *Cours*, about the continuity of the sum of a series of functions. If u_v is a continuous function of x in the neighbourhood of $x=x_o$, s_n r_n $(=s-s_n)$, and s are functions of x, of which s_n is evidently continuous in the neighbourhood of x_o; "then consider the increments which these three functions receive when x increases by an infinitely small quantity α. The increment of s_n will be, for all possible values of n, an indefinitely small quantity; and that of r_n will become insensible at the same time as r_n, if a very considerable value is attributed to n. Consequently the increment of the function s can only be an indefinitely small quantity," and so s is continuous in the neighbourhood of $x=x_o$. This theorem was applied, in § IV of the same chapter, to prove that the sum of a convergent power-series is a continuous function, and this was applied to a proof of the binomial theorem. At this point we arrive at the considerations of Abel, Weierstrass, Stokes, Seidel and Cauchy; but we shall not pursue the subject in this place.[40]

Cauchy gave the same definition of "convergence" as Fourier. Where

$$s_n=u_o+u_1+\dots+u_{n-1}$$

"if", said Cauchy[41], "for values of n which always increase, the sum s_n approaches indefinitely a certain limit s, the series will be called convergent, and the limit in question will be called the sum of the series. On the other hand, if, while n increases indefinitely, the sum s_n does not approach any fixed limit, the series will be divergent and will no longer have a sum."[42] Cauchy then illustrated this distinction by the

[40] Cf. *Archiv der Math. und Phys.* (3), Vol. XIV, pp. 300-301

[41] *Oeuvres* (2), Vol. III, pp. 114-115; cf. Reiff, *op. cit.*, p. 167.

[42] We collect here a few typical instances of the same conception—which depends on the existence of a limit—of *convergence*:

De Morgan (*Elements of Algebra*, 2nd ed., London, 1837, p. 179) defined a "convergent" series as one in which a limit towards which we approximate by continually adding more and more of its terms exists, considered (pp. 159-162) the geometrical series, which has a rational limit, derived (p. 182) the criterion of convergence lim. u_{n+1}/u_n 1, and proved by this criterion that the exponential series is convergent. The result that the exponential series, even though it does not tend to a rational limit, is convergent, is a consequence of the tacit assumption, made in the derivation of the criterion, that a series, every term of which is numerically less than the corresponding term of a convergent series is convergent.

De Morgan said (p. 95): "Every number and fraction has a root of every order, either exact or approximate;" and cautioned (p. 101): "Whenever any new process is introduced, or any new expression, it must be proved, and assumed, that problems involving that process admit *quam proxime*, if not of exact, solutions."

The same doctrine of convergence is also given in De Morgan's *Elementary Illustrations of the Differential and Integral Calculus*, published at London in 1832

geometrical progression 1, x, x^2,..., which converges if $x<1$ and diverges if $x>1$, and proceeded: " By the above principles, in order that the series

$$u_0, u_1, u_2, \ldots, u_n, u_{n+1}, \ldots$$

may be convergent it is necessary and sufficient that increasing values of n make the sum s_n converge indefinitely towards a fixed limit s; in other words, it is necessary and sufficient that, for infinitely great values of n, the sums s_n, s_{n+1}, s_{n+2}...differ from the limit s, and consequently from one another, by indefinitely small quantities. Now, we have

$$s_{n+1}-s_n=u_n\,,\; s_{n+2}-s_n=u_n+u_{n+1},\; etc\,;$$

thus it is necessary that the general term u_n should decrease indefinitely as n increases. But this condition does not suffice; the various sums

$$u_n+u_{n+1},\; u_n+u_{n+1}+u_{n+2}\ldots,$$

taken in any number, must end by constantly having values which are numerically less than any assignable magnitude. Reciprocally, when these various conditions are fulfilled, the convergence of the series is assured." Then Cauchy showed that the harmonic series is not convergent although the general term decreases indefinitely. Further, the series

$$1, \frac{1}{1}, \frac{1}{1\cdot 2}, \ldots, \frac{1}{1\cdot 2\cdots n}, \ldots$$

is convergent, said Cauchy, because each term is less than the corresponding term of a convergent geometrical progression.

This general criterion of convergence occupies a very prominent position in theories of irrational numbers. It cannot, however, be practically used with many series, and accordingly Cauchy gave a series of special criteria, based on the principle of comparison with series known to converge. Here we must pass over the development of the theory of convergence-criteria in the hands of Cauchy and others,[43] but there is one point about the distinction of two kinds of criteria and their efficacy that must be touched upon, as it is very closely connected with the principles of the theory of irrational numbers.

Cauchy considered, in the second section[44] of Chapter VI., series all of whose terms are positive, and obtained, by comparison with the convergent geometrical series, the following criterion of convergence of the series u_0, u_1, u_2,... " Find the limit or limits[45] to which $(u_n)^{1/n}$ con-

(new edition, Chicago 1909, pp. 17-19), and bound up with his large *Differential and Intergral Calculus*, London, 1842. See also pp. 118, 222, 234, of the last-named work.

Nearly all writers follow Cauchy in making the meaning of "convergence" imply the existence of a limit. Cf. *Encykl. der math. Wiss.*, Vol. I, p. 77.

[43] Cf. Reiff, *op. cit.*, pp. 167sqq.; Pringsheim, *Encykl. der math. Wiss.* Vol. I., A. 3, p. 189.

[44] *Oeuvres* (2), Vol. III., pp. 121-128.

[45] For the case of the occurrence of many limits, cf. §VII below.

verges when n is infinite, and let k be the greatest of these limits (or, in other words, the limit of the greatest value of the expression in question). The series converges if $k<1$ and diverges if $k>1$." After the proof of this theorem, Cauchy proceeded: "In a great number of cases, we can determine the value of k by theorem IV of section III of Chapter II. Thus, whenever u_{n+1}/u_n converges to a limit, this limit will be the value of k. From this, Cauchy got a theorem in which a more convenient but less general criterion was stated.

Cauchy[46] emphasized the advantage of the first criterion over the second, for, the upper limit in the first always exists, while that in the second, may not. However, this important distinction has often been overlooked by mathematicians.[47] Nowadays, we follow du Bois-Reymond[48] in distinguishing criteria of "the first" and "second kind."

We have seen that Cauchy defined a "convergent" series as one which has a limit, and then stated a necessary and sufficient condition for convergence, in which no existence of limits was assumed. The necessity of this condition—the proof that if one and only one limit s exists, then, where ϵ is an arbitrarily small positive non-zero number, which is of course rational unless we have already defined the system of real numbers, a whole number n can be found such that

$$| s_n - s_{n+m} | < \epsilon,$$

whatever interger m is—is easily shown. The sufficiency ot the condition—that the last inequality implies the existence of a limit—was asserted, but not proved, by Cauchy. Cauchy's theory seems, in fact, to have been essentially of a geometrical nature.

Before Cauchy, Bernard Bolzano (1817) published a memoir[49] in the course of which he noticed that this criterion stated an important property of series, and went beyond Cauchy in attempting to prove the sufficiency of this criterion for establishing the existence of a limit. This proof failed, and the existence of a limit was, after all, only assumed as a result of an appeal to those of us who "have a correct conception of magnitude." But, at any rate, that point was emphasized which seems to have been universally overlooked by writers of text-books up to quite modern times. It was this oversight which led many persons into the vicious circle of defining a convergent series as one for which a limit exists, and then supposing that certain series defined, by approximation, certain 'irrational' numbers not hitherto considered.

(*To be continued.*)

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46 *Cf.* Pringsheim, *loc. cit.*, p. 81.

47 *Ibid.* Reiff also did not notice this.

48 *Journ. für Math.*, Vol. LXXVI, 1873, p. 61. *Cf.* Pringsheim *loc it.c.*, pp. 80-90

49 Reprinted, with notes, in No. 153 of *Ostwald's Klassiker*.

Isogonal Transformation.

[The substance of this paper was read at a meeting of the Presidency College Mathematical Association. References: Salmon's *Conic Sections*, §§ 55, 297; Salmon's *Higher Plane Curves*, §§ 283, 34-36.]

1. If S, H be two points in the plane of a triangle ABC such that

angle BAS = angle HAC
„ CBS = „ HBA
„ ACS = „ HCB,

then the points S, H are said to be isogonally conjugate with respect to ABC, and their trilinear co-ordinates with respect to ABC are reciprocal.

If each point of a figure is replaced by its isogonal conjugate, a new figure is evidently formed; the transformation so effected is called *isogonal transformation*.

The only *self-conjugate* points are the *in* and *ex* centres of ABC.

The conjugate of any point on a side of ABC is the opposite vertex. A vertex as such, has therefore no *definite* conjugate point, but regarded as lying on a continuous curve its conjugate is a definite point on the opposite side, *viz.*, its intersection with the isogonal conjugate of the tangent at the vertex.

The conjugate of any point on the circumcircle of ABC is at infinity, and the circular points at infinity (being common to the circumcircle and the line at infinity) are conjugate points.

2. The following properties are easily established:—

(i) The transform of a *continuous* curve is a continuous curve.

(ii) Each side of the triangle of reference cuts a curve of the n^{th} degree in n points, and therefore the transform passes through the opposite vertex n times, that is, has a multiple point of the n^{th} order at the vertex.

(iii) The transform of a curve of the n^{th} degree is in general a curve of the $2n^{th}$ degree.

For, if $f(\alpha, \beta, \gamma)=0$ be a curve in trilinear co-ordinates, the transform is seen to be $f(\beta\gamma, \gamma\alpha, \alpha\beta)=0$.

If, however, the original curve has a multiple point of the k^{th} order at A, its equation may be put in the form

$$\alpha^{n-k}\phi_k+\alpha^{n-k-1}\phi_{k+1}+\dots\dots\dots\phi_n=0$$

where ϕ_r is a homogeneous function in β, γ of degree r.

The transform is therefore obtained by replacing α, β, γ by their reciprocals and multiplying by $(\alpha^{n-k}\beta^n\gamma^n)$. It is thus of degree $(2n-k)$.

Hence the degree of the transform is lowered by k, corresponding to a multiple point of the k^{th} order at a vertex on the original curve.

Note.—Though the degree of a curve is generally *doubled* by isogonal transformation, the transform has multiple points of the n^{th} order at each vertex of the triangle of reference, and therefore (by the above rule) a second transformation will give a curve of degree $(2 \cdot 2n - n - n - n) = n$, and is in fact the original curve.

3. *Applications* :—

(1) *From any point two tangents can be drawn to a conic.*

Transforming with respect to a triangle inscribed in the conic, we deduce that *two conics can be described through four points to touch a given straight line.*

(2) *If a triangle circumscribes a parabola its focus lies on the circumcircle of the triangle.*

For, the foci of an inscribed conic are conjugate points; and since one of them is at infinity, the other must lie on the circumcircle.

(3) *If a conic touch the sides of a triangle and its foci lie on a rectangular hyperbola through the vertices of the triangle, the major axis of the conic passes through the circumcentre.* [Mathl. Tripos.]

Since, the circumscribed rectangular hyperbola also passes though the orthocentre, its isogonal transformation is a straight line passing through the circumcentre. Also, the two foci which lie on the rect. hyperbola are themselves conjugate to each other. Hence, the transformed straight line passes through these foci. In other words, the major axis of the in-conic passes through the circumcentre.

(4) *A conic touches four given straight lines. Show that the locus of the foci is a cubic curve passing through the six intersections of the given lines.*

In the first place, it is easily seen that the six intersections are special positions of the foci of the inconic. For, each diagonal of the quadrilateral is a line conic belonging to the system of inscribed conics whose foci are its extremities.

Secondly, the two foci must trace the same locus. For, there is no intrinsic difference in the relation of either focus to the quadrilateral.

Thirdly, the locus cannot be a nodal curve. For, then, there would be more than one conic of the system having the node for a focus.

Hence, transforming with respect to the triangle formed by any three of the given straight lines, the locus is its own conjugate, so that, by § 2 (iii),

$$2n-3=n, \text{ or } n=3,$$

n being the degree of the locus.

4. *Cross-Ratio Properties*:

(1) If (PQRS) be a set of points whose isogonal transformation is (P′Q′R′S′), then the cross-ratios of the pencils A(PQRS), A(P′Q′R′S′) are equal, where A is a vertex of the triangle of reference.

The proof is obvious, since the cross-ratio of a pencil is a function of the sines of the angles between the different rays of the pencil, and isogonal transformation does not affect the angle between the rays joining two points to a vertex of the triangle of reference.

(2) If (P, Q, R, S) be points on a circumconic, their transform (P′Q′R′S′) is a range on a straight line. Thus

$$\begin{aligned} A\,(PQRS) &= A\,(P'Q'R'S') \text{ by (1).} \\ &= (P'Q'R'S') = B\,(P'Q'R'S') \\ &= B\,(PQRS) \text{ by (1).} \end{aligned}$$

Hence we have the constant cross-ratio property of conics. We also learn from the above that *a range on a straight line transforms into an equi-cross range on a conic.*

(3) "*A four-point system of conics cuts a straight line in an involution range.*"

Transform, taking three of the four points to form the triangle of reference, and we have

"*Lines through a fixed point cut a conic in corresponding point of an involution range.*"

(4) The pole and polar property of conics may be stated thus:

"*Tangents at corresponding points of an involution range on a conic et on a straight line.*"

Transforming with respect to an inscribed triangle this becomes:

"*If a three-point system of conics touches a straight line at points of an involution range, the intersection of corresponding members lies on a conic through the three points.*"

As a particular case we may notice the following:

"*Parabolas circumscribing a triangle and having their axes at right angles intersect on a conic.*"

(5) From the involution property of conics follows an interesting involution propetry of nodal cubics.

If a conic passes through the vertex A and cuts the base BC in points distinct from B, C, its isogonal transformation is a nodal cubic having the node at A, and passing through B and C. Now, if transversals $(B\,P_1\,Q_1)$, $(B\,P_2\,Q_2)$...are drawn through B to cut the conic in (P_1Q_1), $(P_2\,Q_2)$,......then $A(P_1P_2......Q_1\;Q_2...)$ is an involution pencil. Transforming, we find that *in the nodal cubic lines through any point B cut the cubic again in pairs of points* $(P_1'Q_1')$ $(P_2'Q_2')$...*such that* $A(P_1'P_2'......Q_1'Q_2'...)$ *is a pencil in involution, where A is the node.*

A. Narasinga Rao.

Note on the Diameter of a Quadrilateral.

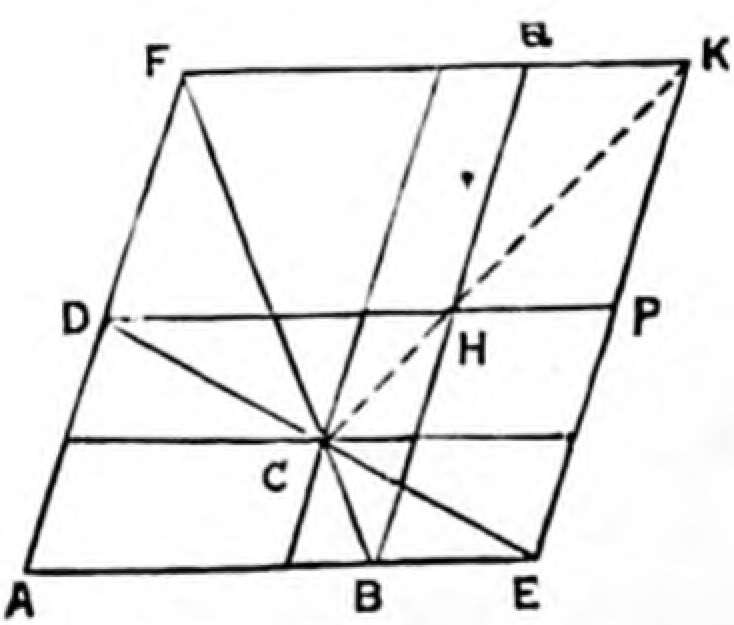
The collinear property of the midpoints of the diagonals of a complete quadrilateral has been proved in Text Books in various ways [*Vide:* (1) Casey: *Sequel to Euclid*, p. 5. (2) Richardson and Ramsey: *Modern Plane Geometry*, p. 72. (3) Askwith: *Pure Geometry*, p. 151. (4) Russell: *Pure Geometry*, pp. 34, 241. (5) Durell: *Plane Geometry*, Pt. I, pp. 118, 119.]

The following elementary proof may be of interest.

Let ABCD be a quadrilateral of which AC, BD, EF are the diagonals. Complete the parallelograms AH, AK as in the figure, and draw parallels through C to AB, AD.

Now in the parallelogram AQ, the complements CA, CQ are equal; and in the parallelogram AP, the complements CA, CP are equal.

$$\therefore \quad \square^{m}\ CP = \square^{m}\ CQ.$$



$\therefore$ CHK is a straight line.

Hence, also, the middle points of AC, AH, AK are in a straight line. But the middle points of AH, AK are the same as the middle points of BD, EF. Therefore the middle points of the diagonals AC, BD, EF are collinear.

M. T. NARANIENGAR.

The Face of the Sky for November and December 1914.

The Sun

enters the winter Solstice on December 22 at 9·54 P.M.

Phases of the Moon.

	November.			December.		
	D.	H.	M.	D.	H.	M.
Full Moon ...	3	5	19	2	11	51
Last Quarter ...	11	5	7	10	5	2
New Moon ...	17	9	32	17	8	5
First Quarter ...	24	7	9	24	1	55

Planets.

Mercury is stationary on November 16 and attains its greatest elongation (19°51′ W) on November 24.

Venus is still an evening star, but approaching the Sun with which it is in inferior conjunction on November 27. It is in conjunction with the Moon on December 15 with Mars on November 22. During these months it is near the boundary between Scorpio and Libra.

Mars is in conjunction with the Sun on December 24 and with the Moon on November 18 and December 17. The planet is in Sagittarius in these months.

Jupiter is in quadrature to the Sun on November 7 and is in conjunction with the Moon on December 21. In these months it is near the boundary between Capricornus and Aquarius.

Saturn is in Taurus. It is in conjunction with the Moon on December 4 and December 31 at 6·51 P.M.

Uranus is in conjunction with the Moon on December 20.

Neptune is stationary on November 3 and is in conjunction with the Moon on November 9 and December 7.

V. RAMESAM.

SOLUTIONS.

Question 339.

(M. BHIMASENA RAU):—The incircle of a triangle ABC touches AC, AB in E, F, and E′, F′ are the midpoints of AC, AB. Show that the radical axis of the circles described on EF′, E′F as diameters passes through the in-Fuerbach point.

Solution by N. Sankara Aiyar, M. A.

Take the medial triangle D′E′F′ as the triangle of reference. Then the equation of the *n. p. c.* of ABC is

$$S\equiv a\beta\gamma+b\gamma\alpha+c\alpha\beta=0.$$

Now the equation of BC is $b\beta+c\gamma=0$.

Hence the equation of the incircle of ABC is

$$\sqrt{(s-a)(b\beta+c\gamma)}\pm\sqrt{(s-b)(c\gamma+a\alpha)}+\sqrt{(s-c)(a\alpha+b\beta)}=0.$$

The common point of these two circles is easily seen to be $\left(\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}.\right)$

The point E is found to be $c(a-c)$, ac, $a(c-a)$.

Similarly F is found to be $b(a-b)$, $a(b-a)$, ab.

If P (l,m,n) be a point on the circle on EF′ as diameter, then PE and PF′ are $\perp r$. That is $m\alpha-l\beta=0$, and $\alpha a(mc-ma-cn)+\beta(a-c)(cn+ab)+c\gamma(al-ma+c)=0$, should be perpendicular.

Thus $am(mc-ma-cn)-l(a-c)(cn+ub)+b\cos A\cdot c(al-ma+c)$
$mc\cos B\,(al-ma+c)-\cos_{|}c\,\{\,m(a-c)(cn+al)-al(mc-ma-cn)\,\}=0.$

Simplifying, we get for the circle on EF′ as diameter

$$c\,S+(a\alpha+b\beta+c\gamma)\,\{\,\beta(a-c)\cos C+\alpha(a-c-c\cos A)\,\}=0.$$

Similarly for the circle on E′F as diameter.

Hence the radical axis of the two circles is

$$\frac{\beta(a-c)\cos C+\alpha(a-c-c\cos A)}{c}=\frac{\gamma(a-b)\cos B+\alpha(a-b-b\cos A)}{b}.$$

i.e., $$a\alpha(b-c)-b\beta(c-a)\cos C-c\gamma(a-b)\cos B=0,$$

which evidently passes through the point $\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$.

Question 431.

(P. V. SESHU AIYAR, B. A., L. T.):—Integrate

$$\int\frac{\sin\theta}{\sin^{2-\frac{1}{n}}n\theta}\,d\theta$$

Solution by N. Durai Rajan.

Write $n\theta=\phi$ and $\frac{1}{n}=m$.

Then $I=m\int \sin m\phi \sin^{m-2}\phi.\, d\phi$

$$=m\int \sin^{m-2}\phi(\cos\phi \sin\overline{m-1}.\phi+\sin\phi \cos\overline{m-1}\phi)d\phi.$$

$$=\frac{m}{m-1}.\sin^{m-2}\phi.\ \sin(m-1)\phi.$$

Replacing ϕ and n

$$I=\frac{1}{1-n}.\ \sin^{\frac{1}{n}-1} n\theta.\sin\left(\frac{1}{n}-1\right)n\theta$$

$$=\frac{1}{1-n}.\frac{\sin(1-n)\theta}{\sin^{1-\frac{1}{n}} n\theta}.$$

Question 505.

(A. C. L. Wilkinson, M.A., F.R.A.S.) :—A point moves along a curve and PQ is drawn making an angle with the tangent at P equal to λ times the angle which the tangent at P makes with the tangent at a fixed point O of the curve. If PQ envelopes a circle, find the equation of the curve and trace the curves for which $\lambda=1$, $\lambda=\frac{1}{2}$, showing that the latter is a unicursal algebraical curve of the fifth degree whose equation is

$$108\, x^4a+x^2(y^3+162ay^2+405a^2y-2916a^3)+y(y^2+26ay-81a^2)^2=0,$$

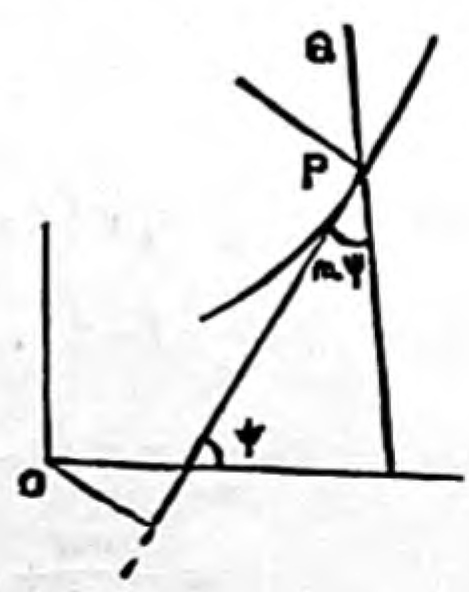
where a is the radius of the given circle, and that the former has an infinite number of parabolic asymptotes given by the equation $y^2=4a+\pi(-x+ar+\pi)$ where r is a positive or negative integer.

Solution by N. Durairajan.

Let

$$x\sin\psi-y\cos\psi=p\equiv f(\psi)$$

be the tangent at P.



Then the normal at P is

$$x\cos\psi+y\sin\psi=\frac{dp}{d\psi}.$$

Since PQ makes an angle $n\psi$ with the tangent at P, its inclination to the x—axis is $(n+1)\psi$. The equation of the PQ is therefore

$$x \sin \psi - y \cos \psi - p + k\left(x \cos \psi + y \sin \psi - \frac{dp}{d\psi}\right) = 0, \qquad \ldots \quad \text{(i)}$$

such that the gradient is $\tan (n+1)\psi$.

$$\therefore \qquad \frac{\sin \psi + k \cos \psi}{\cos \psi - k \sin \psi} = \tan (n+1)\psi.$$

Substituting in (i) we get

$$x \sin (n+1)\psi - y \cos (n+1)\psi = p \cos n\psi + \frac{dp}{d\psi} \sin n\psi.$$

If this touches a circle of radius a the origin being supposed to be the centre of the circle, we have

$$p \cos n\psi + \frac{dp}{d\psi} \sin n\psi = a.$$

Solving this linear equation, we have

$$p(\sin n\psi)^{\frac{1}{n}} = C + \int a(\sin n\psi)^{\frac{1}{n}-1} d\psi.$$

(i) Let $\underline{n=1,}$ then $p \sin \psi = c + a\psi = a(\psi + \alpha)$,
where α is an arbitrary constant.

Taking the typical curve $p = a\psi \operatorname{cosec} \psi$, the tangent and normal are

$$x \sin \psi - y \cos \psi = \frac{a\psi}{\sin \psi},$$

$$x \cos \psi + y \sin \psi = \frac{a \sin \psi - \psi \cos \psi}{\sin^2 \psi}.$$

The co-ordinates of P are given by

$$x = a[\cot \psi + \psi(1 - \cot^2 \psi)]$$
$$y = a[1 - 2\psi \cot \psi].$$

The point P moves to infinity where $\psi = r\pi$.
Now, when ψ approaches $r\pi$, we may write

$$\psi = r\pi + t - \frac{t^3}{3} + \frac{t^5}{5} - \ldots\ldots \left[t = \tan \psi\right]$$

Thus $$\frac{x}{a} = \frac{1}{t} + \left(1 - \frac{1}{t^2}\right)\left(r\pi + t - \frac{t^3}{3} + \ldots\right),$$

$$\frac{y}{a} = 1 - 2\left(r\pi + t - \frac{t^3}{3} + \ldots\right)\frac{1}{t}.$$

$$\therefore \qquad \frac{x}{a} = r\pi + t - \frac{t^3}{3} + \ldots - \frac{n\pi}{t^2} - \frac{1}{t} + \frac{t}{3} - \frac{t^3}{5} + \ldots +$$

$$= -\frac{r\pi}{t^2} + r\pi + \frac{4t}{3} - \ldots\ldots$$

$$\frac{y}{a}=1-2\left(\frac{r\pi}{t}+1-\frac{t^2}{3}+\ldots\right)$$

$$=-1-\frac{2r\pi}{t}+\frac{2t^2}{3}+\ldots\ldots$$

If $\psi\to r\pi$, $t\to 0$, and thus

$\therefore$
$$\frac{x}{a}=-\frac{r\pi}{t^2}+r\pi,$$

$$\frac{y}{a}=-1-\frac{2r\pi}{t}.$$

Eliminating t

$$(y+a)^2=4ar\pi(-x+ar\pi),$$

which is the equation of the parabolic asymptotes.

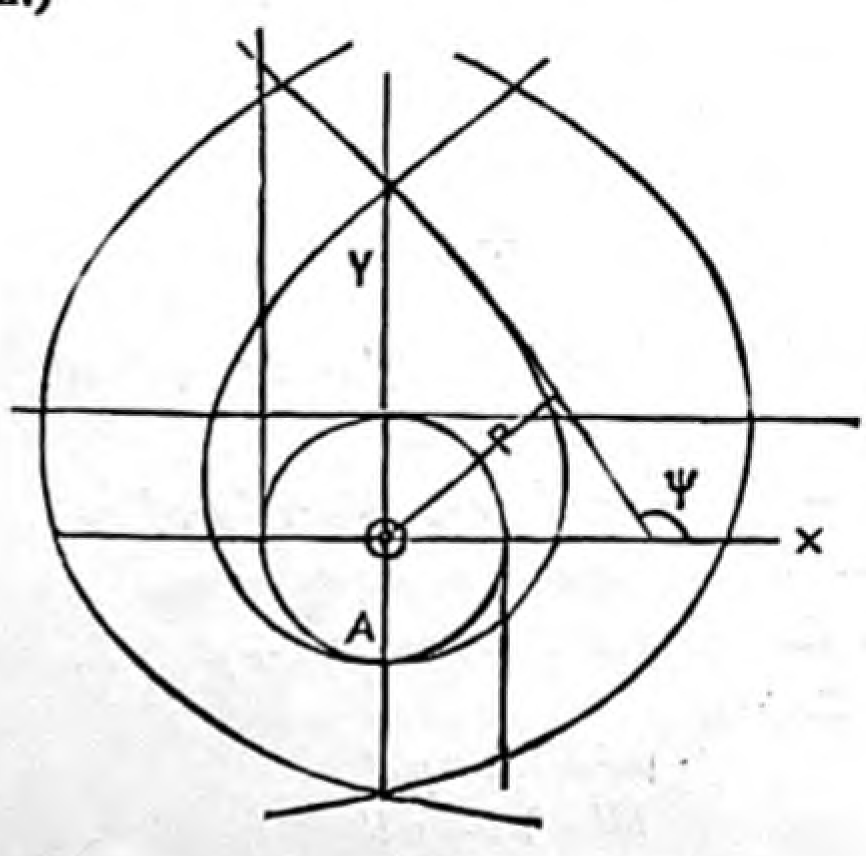
Referring the curve to the point A $(o,-a)$ as origin, the equation becomes $y^2=4ar\pi\ (-x+ar\pi)$.

A tracing of the curve is added. The curve is easily traced by finding out the points where ψ has values

$$o, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi \text{ etc.}$$

The circle is the caustic of the curve for rays parallel to the x-axis

(See Boole's *Differential Equations* for a different method of attacking the problem.)



(ii) Let $n=\frac{1}{2}$.

$$p.\ (\sin\tfrac{1}{2}\psi)^2=C+a\int\sin\tfrac{1}{2}\psi.\ d\psi$$
$$=C-2a\cos\tfrac{1}{2}\psi.$$

Giving C the value $2a$

$$p\sin^2\tfrac{1}{2}\psi=2a(1-\cos\tfrac{1}{2}\psi),$$

$$\therefore \qquad p = a \sec^3 \frac{\psi}{4}.$$

Hence
$$\frac{x}{a}\cos^3 \tfrac{1}{4}\psi = \sin\psi + \tfrac{1}{2}\cos\psi \tan\tfrac{1}{4}\psi,$$

$$\frac{y}{a}\cos^2\tfrac{1}{4}\psi = -\cos\psi + \tfrac{1}{2}\sin\psi \tan\tfrac{1}{4}\psi.$$

Put $\tan\frac{1}{4}\psi = t$, then x and y can be expressed as functions of a single parameter t.

The eliminant is a unicursal curve. I am not able to find out what value of C Prof. Wilkinson has adopted.

Question 514.

(A. C. L. Wilkinson, M. A., F. R. A. S.) :—All conics cutting a rectangular hyperbola orthogonally at all four points of intersection consist of (1) hyperbolas having the axes of the given hyperbola as asymptotes, (2) all conics confocal with the given hyperbola, (3) two sets of ellipses each passing through two fixed points and whose centres lie on one or other of the axes of the hyperbola and whose axes are parallel to the asymptotes of the hyperbola.

Solution by N. Durairajan.

A more general problem is "given a conic, find all the conics cutting the given conic orthogonally at all four points of intersection."

Let the given conic be

$$S \equiv ax^2 + by^2 - 1 = 0,$$

and the orthogonal conic be

$$S' \equiv a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0.$$

The condition for orthogonality is

$$\phi \equiv ax(a'x + h'y + g') + by\,(h'x + b'y + f') = 0.$$

This is true for the four points of intersection.

The conic ϕ is therefore of the form $\lambda S + \mu S' = 0$.

Equating co-efficients

$$\begin{aligned} aa' &= \lambda a + \mu a' \\ bb' &= \lambda b + \mu b' \\ h'(a+b) &= 2\mu h' \\ ag' &= 2\mu g' \\ bf' &= 2\mu f' \\ 0 &= -\lambda + \mu c'. \end{aligned}$$

The third condition gives

$$h' = 0, \text{ or } a + b = 2\mu.$$

The fourth and fifth conditions given

$$g'=0, \text{ or } a=2\mu$$
$$f'=0, \text{ or } b=2\mu.$$

We have thus the following cases :—

(1) $h'=0,\ g'=0,\ f'=0.$

$$S' \text{ becomes } \frac{\lambda a}{a-\mu}x^2+\frac{\lambda b}{b-\mu}y^2+\frac{\lambda}{\mu}=0.$$

$$\therefore \qquad \frac{a}{a-\mu}x^2+\frac{b}{b-\mu}y^2+\frac{1}{\mu}=0.$$

These are conics confocal with $ax^2+by^2=1$.

(2) $h \neq 0$, *i.e.*, $\mu=\frac{1}{2}(a+b)$, $g'=0, f'=0$.

$$\therefore \qquad a'=\frac{\lambda a}{a-\mu}=\frac{2\lambda a}{a-b},\ b'=\frac{2\lambda b}{b-a}$$

Hence S′ becomes

$$2h'xy+\frac{2\lambda}{a-b}\left(ax^2-by^2+\frac{a-b}{a+b}\right)=0,$$

shewing that the conics pass through the intersections of

$$xy=0, \text{ and } ax^2-by^2+\frac{a-b}{a+b}=0.$$

(3) $h'=0,\ g'=0,\ b=2\mu.$

$$\therefore \qquad a'=\frac{2\lambda a}{2a-b},\ b'=2\lambda,\ c'=\frac{2\lambda}{b}.$$

Thus $S'\equiv 2\lambda\left(\dfrac{ax^2}{2a-b}+y^2+\dfrac{1}{b}\right)+2f'y=0.$

(4) $h'=0,\ a=2\mu,\ f'=0$

Similarly, we get

$$x^2+\frac{by^2}{2b-a}+\frac{1}{a}=\lambda x.$$

(5) $h'=0,\ a=2\mu,\ b=2\mu.$

This case is impossible, for, at the points of intersection of the given circle $(a=b)$, the normals to the circle are tangents to the conic: The conic would have four tangents from the centre of the circle, which is not possible.

(6) $h'\neq 0,\ g'=0,\ b=2\mu$, and $2\mu=a+b$.

Here $a=0$., which is contrary to the hypothesis.

So also $h'\neq 0,\ f'=0,\ a=2\mu$.

The condition $\phi=0$ may be identically satisfied by putting.

$$aa'=0,\ bb'=0,\ h'(a+b)=0,\ ag'=0,\ bf'=0.$$

Since a and b are not zero, $a'=0$, $b'=0$, and h' is not zero, S′ being a conic..

$\therefore \qquad a+b=0$ and $g'=0, f=0$.

N. B.—When the given conic is a R. H. $x^2-y^2-\frac{1}{a}=0$, the orthogonal conic is the R.H. $xy=\beta$.

Question 521.

(K. V. Anantanarayana Sastry, B.A.):—If V denote the entire volume of the figure bounded by

$$\left(\frac{x}{a}\right)^{\frac{16}{17}}+\left(\frac{y}{b}\right)^{\frac{16}{17}}+\left(\frac{z}{c}\right)^{\frac{16}{17}}=1,$$

and A the whole area of its trace on the xy—plane, prove that

$$V=\frac{17.Ac}{19.35}.\ B\left(\frac{1}{8},\frac{1}{16}\right).$$

Solution by T. P. Trivedi, M.A., LL.B., R. J. Pocock and P. A. Subramanya Iyer B.A., L.T.

The volume is given by $8\int\int\int dx\ dy\ dz$ where the integral is to extend to all *positive* values of x, y, z subject to the condition

$$\left(\frac{x}{a}\right)^{\frac{16}{17}}+\left(\frac{y}{b}\right)^{\frac{16}{17}}+\left(\frac{z}{c}\right)^{\frac{16}{17}}=1.$$

Employing Dirichlet's formula, the volume

$$V=\frac{8\ abc}{\left(\frac{16}{17}\right)^3}\cdot\frac{\Gamma\left(\frac{17}{16}\right)^3}{\Gamma\left(\frac{51}{16}+1\right)}.$$

The area of the trace on the xy-plane is

$$\frac{4\ ab}{\left(\frac{16}{17}\right)^2}\cdot\frac{\Gamma\left(\frac{17}{16}\right)^2}{\Gamma\left(\frac{34}{16}+1\right)}.$$

$$\therefore \qquad \frac{V}{A}=2c.\frac{17}{16}\ \frac{\Gamma\left(\frac{17}{16}\right)\Gamma\left(\frac{50}{16}\right)}{\Gamma\left(\frac{67}{16}\right)}$$

$$=2c.\frac{\frac{17}{16}.\frac{1}{16}.\frac{34}{16}.\frac{18}{16}.\frac{2}{16}}{\frac{51}{15}.\frac{35}{16}.\frac{19}{16}.\frac{3}{16}}.\frac{\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{1}{16}\right)}{\Gamma\left(\frac{3}{16}\right)}$$

$$=\frac{17c}{19.35}\ B\left(\frac{1}{8},\frac{1}{16}\right).$$

Question 523.

(R. N. APTE M.A., F.R.A.S.) :—Find the value of

$$\iint xyz\left\{1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2\right\}dx\,dy,$$

where $z=c\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{\frac{1}{2}}$, and the integration is over the positive quadrant of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution by T. P. Trivedi M.A., LL.B.

Since $\quad \frac{z}{c}=\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{\frac{1}{2}},\ \frac{z^2}{c^2}=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}.$

$$\frac{z}{c^2}dz=-\frac{x}{a^2}dx,\text{ or }\quad \frac{dz}{dx}=-\frac{x}{z}\frac{c^2}{a^2};$$

Similarly $\quad \frac{dz}{dy}=-\frac{y}{z}\frac{c^2}{b^2}.$

Thus the integral

$$=\iint\frac{xy}{z}\left\{z^2+\frac{c^4}{a^4}x^2+\frac{c^4}{b^4}y^2\right\}dxdy.$$

$$=\iint\frac{xy}{c\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{\frac{1}{2}}}\left\{c^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)+\frac{c^4}{a^4}x^2+\frac{c^4}{b^4}y^2\right\}dxdy$$

Change x into ax and y into by; then $\frac{x^2}{a^2}+\frac{y^2}{b^2}$ becomes x^2+y^2 and $dx\,dy$ becomes $ab\,dxdy$. The integral finally reduces to

$$a^2b^2c\int_0^1\int_0^{\sqrt{1-x^2}}\frac{x}{(1-x^2-y^2)^{\frac{1}{2}}}\left\{(1-x^2-y^2)+\frac{c^2}{a^2}x^2+\frac{c^2}{b^2}y^2\right\}dx\,dy$$

$$=a^2b^2c\int_0^1\left[\int_0^{\sqrt{1-x^2}}\frac{xy}{(1-x^2-y^2)^{\frac{1}{2}}}\left(1-x^2+\frac{c^2}{a^2}x^2\right)dy\right.$$

$$\left.+\int_0^{\sqrt{1-x^2}}\frac{xy^3}{(1-x^2-y^2)}^{\frac{1}{2}}\left(\frac{c^2}{b^2}-1\right)dy\right]dx$$

$$=a^2b^2c\int_0^1\left[-x(1-x^2-y^2)^{\frac{1}{2}}\left((1-x^2+\frac{c^2}{a^2}x^2\right)+\frac{1}{3}x(1-x^2-y^2)^{\frac{3}{2}}\right.$$

$$\left.\times\left(\frac{c^2}{b^2}-1\right)-x(1-x^2)(1-x^2-y^2)^{\frac{1}{2}}\left(\frac{c^2}{b^2}-1\right)\right]_0^{\sqrt{1-x^2}}dx$$

$$=a^2b^2c\int_0^1\left\{x\sqrt{1-x^2}\left(1-x^2+\frac{c^2}{a^2}x^2\right)+\frac{2}{3}x\left(\frac{c^2}{b^2}-1\right)(1-x^2)^{\frac{3}{2}}\right\}dx.$$

Put $x=\sin\theta$;

$$I= a^2b^2c\int_0^{\frac{\pi}{2}}\left\{\sin\theta\cos^2\theta\left(\cos^2\theta+a^2\sin^2\theta\right)+\frac{2}{3}\cos^4\theta\sin\theta\left(\frac{c^2}{b^2}-1\right)d\theta\right\}$$

which gives on reduction

$$\frac{a^2b^2c^3}{15}\left(\frac{2}{a^2}+\frac{2}{b^2}+\frac{1}{c^2}\right).$$

Question 524.

(S. RAMANUJAN) :—Shew that

(i) $\sqrt[3]{\cos\frac{2\pi}{7}}+\sqrt[3]{\cos\frac{\pi}{7}}+\sqrt[3]{\cos\frac{8\pi}{7}}=\sqrt[3]{\frac{5-3\sqrt[3]{7}}{2}}$

(ii) $\sqrt[3]{\cos\frac{2\pi}{9}}+\sqrt[3]{\cos\frac{4\pi}{9}}+\sqrt[3]{\cos\frac{8\pi}{9}}=\sqrt[3]{\frac{3\sqrt[3]{9}-6}{2}}$.

Solution by N. Sankara Aiyar, M. A.

Lemma.—If α, β, γ, be the roots of a given cubic equation to form the equation satisfied by p where $\sqrt[3]{p}=\sqrt[3]{\alpha}+\sqrt[3]{\beta}+\sqrt[3]{\gamma}$.

Cubing we get

$$p=\Sigma\alpha+3(\Sigma\sqrt[3]{\alpha})(\Sigma\sqrt[3]{\alpha\beta})-3\sqrt[3]{\alpha\beta\gamma}.$$

$$\therefore \{p-\Sigma\alpha+3\sqrt[3]{\alpha\beta\gamma}\}^3=27p\{\Sigma\alpha\beta+3\Sigma(\sqrt[3]{\alpha\beta})\Sigma(\sqrt[3]{\alpha^2\beta\gamma})-3\sqrt[3]{\alpha^2\beta^2\gamma^2}\}$$

$$=27p\{\Sigma\alpha\beta+\sqrt[3]{\alpha\beta\gamma}(p-\Sigma\alpha+3\sqrt[3]{\alpha\beta\gamma})-3\sqrt[3]{\alpha^2\beta^2\gamma^2}\}$$

$$\therefore p^3-3p^2(\Sigma\alpha-3\sqrt[3]{\alpha\beta\gamma})+3p(\Sigma\alpha-3\sqrt[3]{\alpha\beta\gamma})^2-(\Sigma\alpha-3\sqrt[3]{\alpha\beta\gamma})^3=27p\{\Sigma\alpha\beta-\Sigma\alpha\sqrt[3]{\alpha\beta\gamma}\}+27p^2\sqrt[3]{\alpha\beta\gamma}.$$

i. e. $$p^3-3p^2(\Sigma\alpha+6\sqrt[3]{\alpha\beta\gamma})+3p\{(\Sigma\alpha-3\sqrt[3]{\alpha\beta\gamma})^2-9\Sigma\alpha\beta+9\Sigma\alpha\sqrt[3]{\alpha\beta\gamma}\}-(\Sigma\alpha-3\sqrt[3]{\alpha\beta\gamma})^3=0.$$

(i) Let $x=\cos\frac{2r\pi}{7}+i\sin\frac{2r\pi}{7}$. The roots of $x^7-1=0$ are the values of $\cos\frac{2r\pi}{7}+i\sin\frac{2r\pi}{7}$.

Substitute $x+\frac{1}{x}=y=2\cos\frac{2\gamma\pi}{7}$. The equation for y is

$y^3+y^2-2y-1=0$, and its roots are

$$2\cos\frac{2\pi}{7},\ 2\cos\frac{4\pi}{7},\ 2\cos\frac{6\pi}{7}\text{ or }2\cos\frac{8\pi}{7}.$$

Substituting in the equation for p the values of $\Sigma\alpha$ $\Sigma\alpha\beta$, and $\alpha\beta\gamma$, we get after reduction

$$p^3-15p^2+75p+64=0,$$

or $$(p-5)^3+189=0.$$

The real value of p is thus $(5-3\sqrt[3]{7})$.

Hence

$$\sqrt[3]{2\cos\frac{2\pi}{7}}+\sqrt[3]{2\cos\frac{4\pi}{7}}+\sqrt[3]{2\cos\frac{8\pi}{7}}=\sqrt[3]{5-3\sqrt[3]{7}}.$$

$$\therefore\ \sqrt[3]{\cos\frac{2\pi}{7}}+\sqrt[3]{\cos\frac{4\pi}{7}}+\sqrt[3]{\cos\frac{8\pi}{7}}=\sqrt[3]{\frac{5-3\sqrt[3]{7}}{2}}.$$

(ii) Again the equation

$$(x^9-1)/(x^3-1)=0,$$

has roots $\cos\frac{2r\pi}{9}+i\sin\frac{2r\pi}{9}$, where $r \not\equiv 0 \pmod 3$.

Substituting $x+\frac{1}{x}=y=2\cos\frac{2r\pi}{9}$, we get the equation for y in the form $y^3-3y+1=0$.

The equation for p reduces as before to

$$p^3+18p^2+108p-27=0,$$

or $$(p+6)^3-248=0.$$

$$\therefore\qquad p=3\sqrt[3]{9}-6.$$

Thus

$$\sqrt[3]{2\cos\frac{2\pi}{9}}+\sqrt[3]{2\cos\frac{4\pi}{9}}+\sqrt[3]{2\cos\frac{8\pi}{9}}=\sqrt[3]{3\sqrt[3]{9}-6}$$

$$\therefore\ \sqrt[3]{\cos\frac{2\pi}{9}}+\sqrt[3]{\cos\frac{4\pi}{6}}+\sqrt[3]{\cos\frac{8\pi}{9}}=\sqrt[3]{\frac{3\sqrt[3]{9}-6}{2}}.$$

Question 525.

(S. Ramanujan) :—Show how to find the square roots of surds of the form $\sqrt[3]{A}+\sqrt[3]{B}$, and hence prove that

$$\sqrt{\sqrt[3]{5}-\sqrt[3]{4}}=\frac{\sqrt[3]{2}+\sqrt[3]{20}-\sqrt[3]{25}}{3}$$

$$\sqrt{\sqrt[3]{28}-\sqrt[3]{27}}=\frac{\sqrt[3]{98}-\sqrt[3]{28}-1}{3}.$$

Solution by N. Sankara Aiyar, M.A.

The square root should evidently be of the form $l\sqrt[3]{p}+m\sqrt[3]{q}+n\sqrt[3]{r}$, where the surds $\sqrt[3]{p^2}$, $\sqrt[3]{pq}$ &c., should be similar in pairs

$\therefore\quad p=a^2b,\ q=b^2,\ r=a$ ... ... (i)

or $\quad p=ab^2,\ q=b^3,\ r=a^2b^2$ ... ... (ii)

or $\quad p=a^3,\ q=ab^2,\ r=a^2b$ ... ... (iii)

Taking (i) we get the square

$$\sqrt[3]{ab^2}(al^2+2mn)+\sqrt[3]{b}(bm^2+2aln)+\sqrt[3]{a^2}(n^2+2\,lmb).$$

One of these coefficients should vanish.

Thus, taking the first example, we get

$$al^2+2mn=0,\ bm^2+2aln=1,$$
$$n^2+2blm=-1 \text{ and } b=5,\ a=2.$$

Hence $$l=\frac{1}{3},\ m=-\frac{1}{3},\ n=\frac{1}{3}.$$

$$\therefore\quad \sqrt{\sqrt[3]{5}-\sqrt[3]{4}}=\frac{\sqrt[3]{20}-\sqrt[3]{25}+\sqrt[3]{2}}{3}.$$

Taking (ii) we get the square in the form

$$\sqrt[3]{a^2b}(bl^2+2bmn)+\sqrt[3]{b}(bm^2+2abln)+\sqrt[3]{ab}(abn^2+2blm).$$

We should make one of these coefficients vanish.

Taking (iii) we get the square in the form

$$l^2a^2+2mnab+\sqrt[3]{ab^2}(an^2+2alm)+\sqrt[3]{a^2b}(bm^2+2aln).$$

Comparing this with the second example above, we get,

$$l^2a^2+2mnab=-3,\ bm^2+2aln=0,$$
$$an^2+2alm=1,\ a=7,\ b=2;$$

or $$l^2a^2+2mnab=-3,\ bm^2+2aln=1,$$
$$an^2+2alm=0,\ a=2,\ b=7.$$

The first gives $l=\frac{1}{21},\ m=\frac{7}{21},\ n=-\frac{7}{21},$

The second gives $l=\frac{1}{6},\ m=-\frac{1}{3},\ n=\frac{1}{3}.$

Both give the same result: *viz.*,

$$\frac{1+\sqrt[3]{28}-\sqrt[3]{98}}{3}.$$

The positive root is therefore

$$\frac{\sqrt[3]{98}-\sqrt[3]{28}-1}{3}.$$

Question 527.

(A. Narasinga Rao.) :—A family of quartics having nodes at A, B, C, pass through four other given points. Shew that only two members of the family touch the circle ABC.

Solution by Proposer.

The isogonal transformation of this quartic family *w.r.t.* Δ ABC is a family of conics passing through the conjugates of the four given points. Also, the isogonal transform of the circle ABC is the line at infinity. Hence the above result follows from the following property of conics:

"Two members of a four point system of conics touch the line at infinity."

Question 528.

(A. N. Raghavachar, M. A.) :—If (n, r) denote the sum of the products r together of the first n natural numbers, find the value of

$$\frac{(2n-1, n)}{\underline{|2n}} - \frac{(2n-2, n)}{\underline{|1}\ \underline{|2n-1}} + \frac{(2n-3, n)}{\underline{|2}\ \underline{|2n-2}} + \dots \frac{(-1)^{n-1}(n, n)}{\underline{|n-1}\ \underline{|n+1}}.$$

Solution by T. P. Trivedi, M.A., LL. B.

We have

$$\frac{[\log(1+x)]^r}{\underline{|r}} = \frac{x^r}{\underline{|r}} - \frac{(r, 1)x^{r+1}}{\underline{|r+1}} + \frac{(r+1, 2)x^{r+2}}{\underline{|r+2}} - \dots$$

(Ex. 11, page 80, Edward's *Diff. Calc.*)

Hence

$$\frac{(2n-1, n)}{\underline{|2n}} = \text{coefficient of } x^{2n} \text{ in } (-1)^n \frac{[\log(1+x)]^n}{\underline{|n}},$$

$$\frac{(2n-2, n)}{\underline{|1}\ \underline{|2n-1}} = \text{coefficient of } x^{2n} \text{ in } (-1)^n \frac{x[\log(1+x)]^{n-1}}{\underline{|1}\ \underline{|n-1}},$$

and so on. Thus the series is equal to the coeff. of x^{2n} in

$$(-1)^n \frac{1}{\underline{|n}} \left\{ [\log(1+x)]^n - \frac{\underline{|n}}{\underline{|1}\ \underline{|n-1}} x [\log(1+x)]^{n-1} + \frac{\underline{|n}}{\underline{|n-2}} x^2 [\log(1+x)]^{n-2} - \text{etc. to } n \text{ terms} \right\}$$

$$= (-1)^n \text{ coeff. of } x^{2n} \text{ in } \frac{1}{\underline{|n}} \left\{ [\log(1+x) - x]^n + (-1)^{n-1} x^n \right\}$$

$$= (-1)^n \text{ coeff. of } x^{2n} \text{ in } \frac{1}{\underline{|n}} \left\{ \left(-\frac{x^2}{2} + \frac{x^3}{3} - \dots \right)^n + (-1)^{n-1} x^n \right\}$$

$$= \frac{1}{\underline{|n}\ 2^n}.$$

Question 531.

(N. Sankara Aiyar, M.A.) :—Integrate

$$\int_0^\infty \frac{\sin xy}{x(1+x^2)^3} dx.$$

Solutions: (1) *by J. M. Bose*, (2) *by K. J. Sanjana, T. P. Trivedi and P. A. Subramanya Iyer, B.A., L.T.*; (3) *by the Proposer.*

(1) Let $I = \int_{-\infty}^{\infty} \frac{\sin xy}{x(1+x^2)^3}$, so that $\frac{\partial I}{\partial y} = \int_{-\infty}^{\infty} \frac{\cos xy}{(1+x^2)^3}$.

Now $\frac{\partial I}{\partial y}$ is the real part of $\frac{e^{yzi}}{(1+z^2)^3}$ taken round a suitable contour in the z-plane.

Consider a contour composed of a semicircle in the upper half of the z-plane together with the real axis from $-\infty$ to $+\infty$.

The function has only one singularity $z=i$ of the third order within the contour.

Let R be the residue at $z=i$.

Then $$R=\underset{z=i}{L}\frac{1}{2!}\left(\frac{d}{dz}\right)^2\left[\frac{e^{yzi}}{(1+z^2)^3}(z-i)^3\right];$$

on differentiating and putting $z=i$, we have

$$R=\frac{-ie^{-y}}{16}(y^2+3y+3),$$

$$\frac{\partial I}{\partial y}=2\pi iR=\frac{\pi}{8}e^{-y}(y^2+3y+3).$$

$$\therefore \qquad I=\frac{\pi}{8}\int e^{-y}(y^2+3y+3)dy+\text{const.}$$

$$=\frac{\pi}{8}\left[8-e^{-y}(y^2+5y+8).\right]$$

Since the integrand is even the reqd. integral $=\frac{1}{2}$I.

Hence $$\int_0^\infty\frac{\sin xy}{x(1+x^2)^3}dx=\frac{\pi}{16}\left[8-e^{-y}(y^2+5y+8).\right]$$

(2) Taking the result

$$\int_0^\infty\frac{\cos xy}{(1+x^2)^3}dx=\frac{\pi}{16}e^{-y}(3+3y+y^2),$$ [*Journal*, Dec. 1913, p. 231]

integrating with respect to y, and determining the constant so that the integral vanishes with y, we obtain

$$\int_0^\infty\frac{\sin xy}{x(1+x^2)^3}dx=C+\frac{\pi}{16}\{-3e^{-y}-(3e^{-y}+ye^{-y})$$
$$-(2e^{-y}+2ye^{-y}+y^2e^{-y})\}$$
$$=\frac{\pi}{16}\left\{8-e^{-y}(8+5y+y^2)\right\}.$$

(3) Let J = the required value of the integral. Then

$$\frac{d^2J}{dy^2}-J=-\int_0^\infty\frac{\sin xy}{x(1+x^2)^2}dx$$
$$=\frac{\pi}{4}ye^{-y}+\frac{\pi}{2}e^{-y}-\frac{\pi}{2}$$

(Vol. VI. p. 26 of J. I. M. S.)

$$\therefore \qquad J=Ae^{y}+Be^{-y}-\frac{\pi}{16}(y^2+5y)e^{-y}+\frac{\pi}{2}.$$

But $J=0$ when $y=0$, and when $y=0$

$$\frac{dJ}{dy}=\int_0^\infty \frac{\cos xy}{(1+x^2)^3}dx=\int_0^\infty \frac{dx}{(1+x^2)^3}=\frac{3\pi}{16}.$$

$\therefore$ $$A+B+\frac{\pi}{2}=0,$$

and $$A-B-\frac{5\pi}{16}=\frac{3\pi}{16},$$

$\therefore$ $$A=0,\ B=-\frac{\pi}{2}.$$

Hence $$J=\frac{\pi}{2}-e^{-y}(y^2+5y+8)\frac{\pi}{16}$$

Question 536.

(K. Appukuttan Erady M. A.) :—Prove that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} ue^{-u'}\,dxdy=\frac{\pi}{2}.\quad \frac{ab'+a'b-2hh'}{(a'b'-h'^2)^{\frac{3}{2}}},$$

where $u\equiv ax^2+2hxy+by^2$; $u\equiv a'x^2+2h'xy+b'y^2$.

Solution (1) *by R. Vythynathaswamy aud* (2) *by R. J. Pocock, B.A., B.Sc , F.R.A.S.*

(1) Consider the pairs of lines represented by

$$ax^2+2h\,xy+by^2=0 \quad \dots \quad \dots \quad \dots (1)$$
$$a'x^2+2h'xy+b'y^2=0. \quad \dots \quad \dots \quad \dots (2)$$

Transform the axes so as to coincide with the bisectors of the angles between the lines (2). Let the transformed equations be

$$Ax^2+2H\,xy+By^2=0 \quad \dots \quad \dots \quad \dots (1^1)$$
$$Lx^2+My^2=0 \quad \dots \quad \dots \quad \dots (2^1)$$

Then the given integral reduces to

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(Ax^2+2H\,xy+By^2\right)e^{-Lx^2,-My^2}\,dx\,dy,$$

since $$\int_{-\infty}^{\infty} xe^{-Kx^2}dx=0.$$

The above $=\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(Ax^2+By^2)\,e^{-L.x^2}e^{-My^2}dxdy.$

Noting that, $\displaystyle\int_{-\infty}^{\infty} e^{-ax^2}dx=2\int_{-0}^{\infty} e^{-ax^2}dx$, the above

$$=\int_{-\infty}^{\infty}\left(\frac{A}{2}\frac{\sqrt{\pi}}{L^{\frac{3}{2}}}\,e^{-My^2}+\frac{\sqrt{\pi}}{L}By^2.\,e^{-My^2}\right)dy$$

$$=\sqrt{\frac{\pi}{M}}\,\frac{A}{2}\cdot\frac{\sqrt{\pi}}{L^{\frac{3}{2}}}+\sqrt{\frac{\pi}{L}}\cdot\frac{\sqrt{\pi}}{M^{\frac{3}{2}}}\cdot\frac{B}{2}$$

$$=\frac{\pi}{2}\left(\frac{AM+BL}{(LM)^{\frac{3}{2}}}\right)$$

And by invariants

$$AM+BL=ab'+a'b-2hh',$$

and

$$LM=a'b'-h'^2,$$

whence the result reduces to

$$\frac{\pi}{2}\,\frac{(ab'+a'b-2hh')}{(a'b'-h'^2)^{\frac{3}{2}}}.$$

(2) By the substitution $X\sqrt{a'}=a'x+h'y$; $Y\sqrt{a'}=y\sqrt{a'b'-h'^2}$, u' becomes X^2+Y^2, and u becomes

$$\frac{a}{a'}\cdot X^2+\frac{2X\cdot Ya}{\sqrt{a'b'-h'^2}}\cdot\left(h-\frac{h'}{a'}\right)+\frac{Y^2}{(a'b'-h'^2)}\cdot\left\{\frac{ah'^2}{a'}-2\,hh+a'b\right\}.$$

Also $dxdy$ becomes $\dfrac{dXdY}{\sqrt{a'b'-h'^2}}$ (Williamson's *Integral*, § 223).

Now
$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+}XYe^{-X^2-Y^2}dX\,dY=$$

$$=\int_{-\infty}^{+\infty}Xe^{-X^2}dX\int_{-\infty}^{+\infty}Ye^{-Y^2}dY=0,$$

since the integrals vanish at both limits.

Also

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}X^2e^{-X^2-Y^2}dX\,dY=\int_{-\infty}^{+\infty}X^2e^{-X^2}dX=\int_{-\infty}^{+\infty}e^{-Y^2}dY$$

$$=4\int_{0}^{+\infty}X^2e^{-X^2}X\int_{0}^{+\infty}e^{-Y^2}dY,$$

$$=2\int_{0}^{+\infty}z^{\frac{1}{2}}e^{-z}dz\int_{0}^{+\infty}e^{-Y^2}dY,$$

writing $X^2=z$.

The first integral$=\Gamma(\frac{3}{2})=\dfrac{\sqrt{\pi}}{2}$, the second also$=\dfrac{\sqrt{\pi}}{2}$.

The given integral

$$=\frac{1}{\sqrt{a'b'-h'^2}}\left\{\frac{a}{a'}+\left(\frac{ah'^2}{a}-2hh'+a'b\right)\frac{1}{(a'b'-h'^2)}\right\}\frac{\pi}{2},$$

$$=\frac{\pi}{2}\,\frac{ab'+a'b-2hh'}{(a'b'-h'^2)^{\frac{3}{2}}}.$$

Question 537.

(K. J. Sanjana, M.A.) :—Prove that

$$\frac{1}{2}\cdot\frac{x^2}{2}-\frac{1.3}{2.4}\,\frac{x^4}{4}+\frac{1.3.5}{2.4.6}\,\frac{x^6}{6}-\ldots\ldots$$

$$=\frac{1}{4}\log(1+x^2)+\frac{1}{8}\log(1+x_1^2)+\frac{1}{16}\log(1+x_2^2)+\ldots\ldots$$

where $x_1^2=\dfrac{x^4}{4(x^2+1)}$, $x_2^2=\dfrac{x_1^4}{4(x_1^2+1)}\ldots\ldots;$

and that

$$\frac{1}{2}\,\frac{x^2}{2}+\frac{1.3}{2.4}\,\frac{x^4}{4}+\frac{1.3.5}{2.4.6}\,\frac{x^6}{6}+\ldots$$

$$=-\tfrac{1}{4}\log(1-x^2)-\tfrac{1}{8}\log(1+x_1^2)-\tfrac{1}{16}\log(1+x_2^2)\ldots$$

where $x_1^2=\dfrac{x^4}{4(1-x^2)}$, $x_2^2=\dfrac{x_1^4}{4(1+x_1^2)},\ldots\ldots$

In the first equality x is any real quantity ; in the second, any real quantity <1.

Solution by T. P. Trivedi, M.A., L L.B.

We have

$$\int_0^{\frac{\pi}{2}}\log(1+n\cos^2\theta)d\theta=\frac{\pi}{4}\log\left\{(1+n)(1+n_1)^{\frac{1}{2}}(1+n_2)^{\frac{1}{4}}\ldots\ldots\ldots\right\},$$

where n, n_1, n_2 etc., are connected by the relation $n_{r+1}=\dfrac{n_r}{4(n_r+1)}$.

(Ex 15, page 75, Todhunter's *Integ. Calc.*)

Taking $n=x^2$, we find that

$$\tfrac{1}{4}\log(1+x^2)+\tfrac{1}{8}\log(1+x_1^2)+\ldots\ldots=\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\log(1+x^2\cos^2\theta)d\theta$$

$$=\frac{1}{\pi}\left\{\int_0^{\frac{1}{2}\pi}\left(x^2\cos^2\theta-\frac{x^4\cos^4\theta}{2}+\ldots\ldots\right)d\theta\right.$$

$$=\frac{1}{2}\cdot\frac{x^2}{2}-\frac{1.3}{2.4x^4}+\frac{1.3.5}{2.4.6}\,\frac{x^6}{6}-\ldots\ldots$$

= the left side of the first equality.

The second result follows by changing x into xi or by reducing $\int_0^{\frac{1}{2}\pi}\log(1-x^2\cos^2\theta)d\theta$, as in the first case.

QUESTIONS FOR SOLUTION.

575. (S. Narayana Aiyar, M.A.) :—If B_1, B_3, B_5, B_7 etc., are the Bernoulli's numbers $\frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{1}{30}$, etc., show that

$$B_1-B_3+B_5-B_7+B_9-B_{11}+\ldots\ldots\text{to }\infty=\frac{\pi^2-9}{6}.$$

576. (S. Narayana Aiyar, M.A.) :—Shew that

$$e^x=\frac{1+x\cdot\frac{a}{b}+\frac{x^2}{1\cdot2}\cdot\frac{a(a+1)}{b(b+1)}+\ldots\ldots\ldots\text{to }\infty}{1+x\cdot\frac{a-b}{b}+\frac{x^2}{1\cdot2}\cdot\frac{(a-b)(a-b-1)}{b(b+1)}+\ldots\ldots\ldots\text{to }\infty}$$

577. (K. Appukuttan Erady, M.A.) :—If S, S′ are the semi-sums of opposite angles of a spherical quadrilateral, prove that

$$\cos(S-S')\sin\tfrac{1}{2}a\sin\tfrac{1}{2}b\sin\tfrac{1}{2}c\sin\tfrac{1}{2}d-\cos(S+S')\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b\cos\tfrac{1}{2}c\cos\tfrac{1}{2}d$$
$$=\tfrac{1}{4}\Sigma(\cos a),$$

a, b, c, d being the sides of the quadrilateral.

578. (N. P. Pandya) :—Construct a triangle ABC, having given the side BC, the distance between the circumcentre and the symmedian point, and also the distance between the incentre and the excentre.

579. (S. Narayanan, B.A., L.T.) :—With the three vertices of a triangle ABC as centres, circles are described to touch the opposite sides. Shew that the three common chords of these circles with the circumcircle from a triangle which is in perspective with the original iangle.

580. (R. Vythynathaswamy) :—Shew that the algebraical factors of the determinant

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 & l^2 \\ 1 & c^2 & 0 & a^2 & m^2 \\ 1 & b^2 & a^2 & 0 & n^2 \\ 1 & l^2 & m^2 & n^2 & 0 \end{vmatrix}$$

are given by

$$\{\pm f(a,m,n)\pm f(b,n,l)\pm f(c,l,m)-f(a,b,c)\},$$

the ambiguous signs being taken all positive or two positive, where

$$\{f(x, y, z)\}^2 = \begin{vmatrix} 0 & x & y & z \\ x & 0 & z & y \\ y & z & 0 & x \\ z & y & x & 0 \end{vmatrix}$$

581. (R. VYTHYNATHASAWMY) :—Shew that

(1) ${}_1C_0 - {}_5C_2 + {}_9C_4 - {}_{13}C_6 \ldots\ldots$ to $\infty = \frac{1}{2}\sin\left(\frac{1}{2}\tan^{-1}4\right)$,

(2) $-{}_3C_2 + {}_7C_4 - {}_{10}C_6 \ldots\ldots$ to $\infty = \frac{1}{2}\cos\left(\frac{1}{2}\tan^{-1}4\right)$.

582. (J. C. SWAMINARAYAN, M.A.) :—Prove the identities

$$\tan^{-1}x - \tan^{-1}\frac{x}{3} + \tan^{-1}\frac{x}{5} - \tan^{-1}\frac{x}{7} + \ldots\ldots = \frac{1}{2}\,\mathrm{gd}\,\frac{\pi}{2}x,$$

and

$$[\tanh^{-1}x - \tanh^{-1}\frac{x}{3} + \tanh^{-1}\frac{x}{5} - \tanh^{-1}\frac{x}{7} + \ldots\ldots\ldots = \frac{1}{3}\,\mathrm{gd}^{-1}\left(\frac{\pi}{2}x\right).$$

583.—(J. C. SWAMINARAYAN, M.A.) :—Shew that

$$\int_0^{\frac{\sqrt{5}-1}{2}} \frac{\log(1+x)}{x}\,dx = \frac{\pi^2}{15} - \frac{1}{2}\left\{\log\frac{\sqrt{5}-1}{2}\right\}^2$$

and

$$\int_0^{\frac{\sqrt{5}-1}{2}} \frac{\log(1-x)}{x}\,dx = \left\{\log\frac{\sqrt{5}-1}{2}\right\}^2 - \frac{\pi^2}{10}.$$

(Suggested by Question 572 of Mr. Bhimansena Rao.)

584. (S. RAMANUJAN) :—Examine the correctness of the following results :—

$$\text{(i)}\quad 1 + \frac{x}{1-x} + \frac{x^4}{(1-x)(1-x^2)} + \frac{x^9}{(1-x)(1-x^2)(1-x^3)} + \ldots\ldots$$
$$= \frac{1}{(1-x)(1-x^6)(1-x^{11})(1-x^{16})} \cdots \times \frac{1}{(1-x^4)(1-x^9)(1-x^{14})} \cdots\cdots$$

Here 1, 4, 9...on the left side are the squares of natural numbers. while 1, 6, 11, 16...and 4, 9, 14,...on the right side are numbers in A.P. with 5 for common difference.

$$\text{(ii)}\quad 1 + \frac{x^2}{1-x} + \frac{x^6}{(1-x)(1-x^2)} + \frac{x^{12}}{(1-x)(1-x^2)(1-x^3)} + \ldots\ldots$$
$$= \frac{1}{1-x^2)(1-x^7)(1-x^{12})} \cdots \times \frac{1}{(1-x^3)(1-x^8)(1-x^{13})} \cdots\cdots$$

Here the n^{th} term of the series 2, 6, 12...is $n(n+1)$, and 2, 7, 12...; 3, 8, 13,...increase by 5.

585. (K. V. Ananthanarayana Sastri, B.A.) :—Prove that

$$\int_0^{\frac{\pi}{2}} \tan^{-1}(a \sin \phi)\, d\phi = \frac{\pi^2}{4} + \log m \log\left(\frac{1-m}{1+m}\right) - 2\sum_0^{\infty} \frac{m^{2r+1}}{(2r+1)^2},$$

where $m=(1+a^2)^{\frac{1}{2}}-a$.

586. (K. V. Anantanarayana Sastri, B.A.) :—Prove that the length of the first negative pedal with respect to the origin of the loop of the Folium of Descartes $x^3+y^3-3axy=0$ is equal to $6a-a\{\pi-\sqrt{2}\log(1+\sqrt{2})\}$.

587. (T. P. Trivedi, M. A., LL.B.) :—If the centre of pressure of a triangle immersed in a liquid coincides with the symmedian point prove that the depths of the angular points are in the ratios

$$3a^2-b^2-c^2;\ 3b^2-c^2-a^2;\ 3c^2-a^2-b^2.$$

588. (T. P. Trivedi, M.A., LL.B.) :—If S_n denote the series

$1-\frac{1}{3^3}+\frac{1}{5^3}-$etc., to n terms, prove that

$$\sum_{n=1}^{n=\infty} \frac{(-1)^{n-1} S_n}{(2n+1)^3} = \frac{\pi^6}{15\cdot 2^{11}}$$

589. (S. Krishnasawmi Aiyangar) :— ABCD is a quadrilateral and its diagonals meet at right angles at S. With S as focus two conics α, β are described one circumscribing the quadrilateral and the other inscribed in it. Show that

1. the two conics have one directrix identical,
2. the envelope of polars with respect to β of points on α is a conic having one focus and one directrix common with those of α and β.
3. the lines joining the points of contact of β with the sides of the quadrilateral meet two by two at the vertices of the harmonic triangle of the quadrilateral.

590. (A. A. Krishnawami Aiyangar) :—Solve completely :

$$(1-4x^2)^2\frac{d^2y}{dx^2}+(1-4x^2)\frac{dy}{dx}+(2x+1)(2x+3)y=0.$$

591. (A. A. Krishnaswami Aiyangar) :—Two circles intersect at A and B. The tangents at the extremities of a double chord through A meet in X. Show that XY perpendicular to BX envelops a circle.

List of Periodicals Received.

(From 16th July to 15th September 1914.)

1. The American Journal of Mathematics, Vol. 36, No. 3, July 1914.
2. The Astrophysical Journal, Vol. 39, No. 5, June 1914.
3. Bulletin of the American Mathematical Society, Vol. 20, No. 10, July 1914.
4. Bulletin des Sciences Mathematiques, Vol. 38, July 1914.
5. Crelle's Journal, Vol. 144, No. 4, June 1914.
6. The Educational Times, July and August 1914, (6 copies).
7. L'Education Mathematique, Vol. 16, Nos. 18, 19 and 20.
8. L'Intermediaire des Mathematiciens, Vol. 21, No. 6, June 1914.
9. Journal de Mathematiques Elementaires, Vol. 38, Nos. 18, 19 and 20.
10. Mathematical Gazette, Vol. 7, No. 112, July 1914, (4 copies).
11. Mathematical Reprints from Educational Times, Vol. 25, (2 copies).
12. Mathematics Teacher, Vol. 6, No. 4, June 1914.
13. Mathematische Annalen, Vol. 75, No. 4, July 1914.
14. Mathesis, Vol. 4, June, July and August 1914.
15. Messenger of Mathematics, Vol. 44, Nos. 1 and 2, May and June 1914.
16. Monthly Notices of the Royal Astronomical Society, Vol. 74, No. 8, June 1914.
17. Philosophical Magazine, Vol. 28, Nos. 163, and 164, July and August 1914.
18. Proceedings of the London Mathematical Society, Vol. 13, No. 5, July 1914.
19. Proceedings of the Royal Society of London, Vol. 90, Nos. 619, 620 and 621, July and August 1914.
20. Revue de Mathematics Speciales, Vol. 24, No. 10, July 1914.
21. Transactions of the American Mathematical Society, Vol. 15, No. 3, July 1914.
22. Transactions of the Royal Society, of London, Vol. 214, No. 514.
23. The Tohoku Mathematical Journal, Vol. 5, Nos. 3 and 4, June 1914.
24. Nature, Vol. 93, Nos. 2334, July 1914.
25. Rendiconti del Circolo Matematico, Di Palermo, Vol. 38, No. 1.
26. The American Mathematical Monthly, Vol. 21, No. 6, June 1914.

The Indian Mathematical Society.

(Founded in 1907 for the Advancement of Mathematical Study, and Research in India.)

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THE JOURNAL

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Vol. VI.] DECEMBER 1914. [No. 6.

PROGRESS REPORT.

Mr. R. Krishnaswami Aiyar, B.A.—Sub-Assistant Inspector of Schools, Anantapur Range, Anantapur—has been elected a member at the concessional rate.

2. In accordance with Art. VIII (*c*) of the constitution of the Society, the constituted Committee for the ensuing two years has been nominated as under—

1. Mr. E. W. Middlemast, M.A.—(President);

2. The Hon'ble Mr. R. P. Paranjpye, M.A., B.Sc.,—(Librarian);

3. The Rev. C. P. Pollard—(Treasurer, but Mr. S. Narayana Aiyar will continue to do his duties during his furlough);

4. Mr. R. N. Apte, M.A., L.L.B., F.R.A.S.

5. Mr. Balakram, M.A., I.C.S.

* 6. Rao Bahadur K. Balaram Iyer, B.A.

7. Mr. Gopal Singh Chowla, M.A.

8. Mr. D. D. Kapadia, M.A., B.Sc.

† 9. Mr. M. P. Khareghat, I.C.S. (Retd.)

10. Mr. M. T. Naraniengar, M.A.

11. Mr. S. Narayana Aiyar, M.A., F.S.S.

12. Mr. P. V. Seshu Aiyar, B.A., L.T.

13. Mr. A. C. L. Wilkinson, M.A., F.R.A.S.

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* Mr. R. B. Balaram Iyer will continue till Rev. Pollard returns.

† Mr. Khareghat retires from the committee when Mr. Ramaswami Aiyar accepts his seat.

POONA,
30th November 1914.

D. D. KAPADIA,
Hony. Joint Secretary.

The Theory of Irrational Numbers, Part I.

By Phillip E. B. Jourdain.

(*Continued from page* 175.)

V.

Bolzano was concerned to prove the theorem in the theory of equations that, between two values of the unknown which give the function considered different signs, one real root at least of the equation must lie. He gave a most acute criticism of the previous proofs of this theorem. That proof was to be rejected which some people have carried out by using a conception of the "continuity" of a function, with an admixture of the conceptions of time and motion. "If the two functions," they said "vary according to the law of continuunity and, for $x=a$, $f(a)<\phi(a)$, but for $x=b$, $f(b)>\phi(b)$ there must be a value u between a and b for which $f(u)=\phi(u)$." The picture which is at once called up of two bodies in motion cannot be regarded as more than a mere example which does not prove the theorem itself, but must be proved by it According to Bolzano an "incorrect conception of continuity" is here used. "According to a correct explanation, we understand by the phrase that a function $f(x)$ varies for all values of x which lie within or without certain limits according to the law of continuity, only so much that, if x is any such value, the difference $f(x+\omega)-f(x)$ can be made less than any given magnitude, if we can take ω as small as we wish." The supposition that a continuous function can never reach a higher value without passing through all lower ones is a theorem which follows, indeed, from the above conception, but can only be proved by the theorem proved in Bolzano's memoir.

The theorem to be proved evidently rests on the more general theorem that, if two continuous functions $f(x)$ and $\phi(x)$ are such that, for $x=a$, $f(a)<\phi(a)$, and, for $x=b$, $f(b)>\phi(b)$, there must be a value of x, lying between a and b for which $f(x)=\phi(x)$. If $f(a)<\phi(a)$, then, by the law of continuity, we also have $f(a+i)<\phi(a+i)$, if we take i small enough. Thus the property of making f smaller than ϕ is possessed by all values of i which are less than a given one. However, this property is not possessed by all values of i, for $f(b)>\phi(b)$. Now, we have the theorem that, when certain property M is possessed by all values of a variable i which are less than a given one and not by all, there is a greatest value u such that all values of i which are less than u possess the property M. For this value we must have $f(a+u)=\phi(a+u)$. The theorem just mentioned was proved "by showing that those values of i

of which it can be asserted that all lesser ones possess the property M, and those of which this cannot be asserted can be brought as near to one another as we wish. Hence results, for everyone who has a correct conception of magnitude, that the thought of an i which is the greatest of those of which it may be said that all less than it is possessed of the property M, is the thought of an *actual* magnitude".[1]

Bolzano then began his formal demonstration by pointing out that there are infinite series— a geometrical progression whose base is a proper fraction— such that, if only n is chosen large enough, what may be denoted by $s_{n+m}(x)$ differs, however great m may be, from $s_n(x)$ by a magnitude less than any given one however small.[2] "*A fortiori* this must hold of a series whose terms decrease more rapidly than those of a falling geometrical progression".[3] If, then, this condition—that, if ϵ is given to be as small as we please, then, if n is large enough, the difference between s_n and any later sum s_{n+m} is less than ϵ—is fulfilled for an infinite series, "there is a certain stationary magnitude (*beständge Grösse*), and indeed only one, which the terms of this series of sums approximate ever more closely, and to which they can come as close as we please if only we continue the series far enough."[4]

Bolzano's proof,[5] which is a very essential element in the proof[6] of the main theorem that, if a property M is possessed not by all values of a variable magnitude x but by all which are less than a certain u, there is a magnitude U which is the greatest of those of which it can be asserted that all x's less than it have the property M, is as follows: "The supposition that a magnitude X, to which the terms of the series of sums

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[1] *Ibid.*, p. 14.

[2] *Ibid.*, pp. 19-20. We may notice that Bolzano considered series whose terms depend on a variable x. Thus, what we denote, following Cauchy, by s, he denoted by $F''x$, (*ibid.*, p. 39). That, as soon as we consider the sum of n terms of a series as depending on another variable (x) besides n, we get the possibility that, though $s_n(x)$ may tend to a definite limit for each x in the domain considered, there may be no number which is not surpassed by sums of n terms calculated for certain x's, was not noticed by Bolzano. Gauss (cf. M. Pasch, *Veranderliche und Funktion*, Leipzig and Berlin, 1914, pp. 141-142, where a further development of Gauss's conceptions of upper and lower limit was mentioned), in his inaugural dissertation of 1799, remarked, as we have seen above (§ II), that if X can take the value S and not the value U, it does not follow that between S and U there must be a value T which X can reach but not surpass, for it is possible that there is a limit between S and U to which X can approach as closely as we wish without reaching it.

[3] *Ibid.*, p. 20.

[4] *Ibid.*, p. 21.

[5] *Ibid.*, pp. 21-23.

[6] *Ibid.*, pp. 25-39; cf. esp. pp. 27-28.

approach as closely as we wish as the series is continued certainly contains nothing impossible when we do not presuppose that this magnitude is single and invariable. For, if it is to be a magnitude which may vary, we shall be able to take it at every moment so that it comes very near to the term $s_n(x)$, with which it is compared, or indeed coincides completely with it. But that the supposition of a constant magnitude which has the property of approximation to the terms of our series contains no impossibility, follows from the fact that it is possible on this supposition to determine this magnitude as accurately as we wish. For suppose that we wish to determine X so accurately that the difference between the supposed value and the true value of X does not surpass a given magnitude d which is as small as we wish, we have only to seek a term $F_n(x)$ of the given series such that any following term $F_{n+r}(x)$ differs from it by less than $\pm d$. By the supposition, there must be such a $F(x)$. Now I say that the value of $F_n(x)$ differs from the true value of the magnitude X by at most $\pm d$. For if, with any n, r is increased at wish, the difference $X-F_{r+1}(x)=\pm\omega$ can become as small as we wish. But the difference $F_n(x)-F_{n+r}(x)$ remains less than $\pm d$, however great we take r. Thus the difference

$$X-F_n(x)=X-F_{n+r}(x)-[F_n(x)-F_{n+r}(x)]$$

must remain less than $\pm(d+\omega)$. But since this is a constant magnitude with the same n, and ω can be made as small as we wish by increasing r, $X-F_n(x)$ must be equal to or less than $\pm d$. For, if it were greater and, for example, equal to $\pm(d+e)$, then the relation $d+e<d+\omega$, that is to say $e<\omega$, could not possibly subsist if ω is diminished more and more. The true value of X therefore differs from the value of the term $F_n(x)$ by d, and consequently, since we can take d as small as we wish we can determine the true value as accurately as we wish. There is therefore a real magnitude, to which the terms of the series referred to approximate as nearly as we wish if we continue it far enough. But there is only one such magnitude. For if we supposed that besides X there were another constant magnitude Y to which the terms of the series approximate as much as we wish if the series is continued far enough, then the differences $X-F_{n+r}(x)$, and $Y-F_{n+r}(x)$ must be capable of becoming as small as we wish if r is allowed to become great enough. Therefore the same must hold of their own difference $X-Y$; and this is impossible, if X and Y are to be constant magnitudes, unless we suppose that $X=Y$.

VI.

Thus we see that Bolzano did not presuppose the existence of a limit. In fact, his procedure was practically the same as that of Meray

who defined a sequence to be "convergent" if the criterion above referred to is satisfied. Later on, we will consider Meray's theory in detail; but at present we must point out that there is no logical evidence that, in general, such a "convergent" series has a limit at all. Suppose that the set of rational numbers is defined. It is not difficult to give an example of a series of rational terms which is "convergent" and yet has no rational limit. Perhaps nearly everybody would admit, on the ground of unanalyzed ideas of a geometrical "continuum," that, if we represent the various rational sums s_n, s_{n+1}, s_{n+2}, ...of the series in question by points at the corresponding distances on a straight line, these points would define a limiting point. But we can only affirm that the corresponding series has a numerical limit if we assume that to every length corresponds a number, which consequently is not always rational. But this is to assume the existence of irrational numbers.

Proofs of the existence of a limit in the case that what may be called "the Bolzano-Cauchy criterion of convergence" is satisfied were attempted by Hankel[7] and Stolz[8]. If $f(x)$ approaches a limit as x tends to a, we can easily show that, for any possible σ, a possible ϵ exists such that, if $0<\delta<\epsilon$, then

$$[f(a+\epsilon)-f(a+\delta)]<\sigma.$$

The converse proposition that this condition suffices to prove the existence of a limit, must, in Hankel's[9] view, be proved and this can only happen by investigation of the conditions under which there is no limit as x approaches a. In this case we can never choose σ so small that, if $0<\delta<\epsilon$, then

$$[f(a+\epsilon)-f(a+\delta)]<\sigma.$$

Thus, if $f(x)$ does not approach any limit, our supposition is contradicted. Consequently the inverse holds.

Stolz's[10] proof by means of the "limits of indetermination" of $F_n(x)$ introduced by P. du Bois-Reymond, is only apparent, since it presupposes the existence of a limit of indetermination just as Bolzano's theorem pre-supposes the definition of a real number.[11]

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7 Article "Grenze" in Ersch and Gruber's *Allg Encyclopadie*, 1. Sekt., Vol. XC. 1871, p. 193. Hankel, in § 19 of this article, was the first mathematician to call attention to the merits of Bolzano's mathematical work.

8 On p. 260 of his article "B. Bolzano's Bedeutung in der Geschichte de Infinitesimalrechnung", *Maths. Ann.*, Vol. XVIII, 1861, pp. 255-279. See *Ostwald's Klassiker*, No. 153, p. 42.

9 *Ibid.*, p. 107. cf. p. 95.

10 *Loc. cit.* p. 260.

11 *Ostwald's Klassiker*, No. 153, p. 42.

VII

Du Bois-Reymond's treatment of the questions lying at the basis of the theory of convergence is very instructive. He noticed[12] that the principles which held for the convergence of integrals were also fundamental for the convergence of all the series known to him, and that besides this, they serve to prove the then unproved convergence of certain of the series used in mathematical physics and which represent arbitrary functions. These principles can be combined into a small group of theorems about the sum

$$u_1v_1+u_2v_2+\ldots\ldots,$$

and correspond to the cases where the sum $u_1+u_2+\ldots$is (1) absolutely convergent, (2) conditionally convergent, (3) indeterminate. Putting $u_1+u_2+\ldots+u_n=U_n$ and $\lim_{n=\infty} U_n=U$, du Bois-Reymond distinguished four cases:

1. U is finite and determined, as in the case of the series $1-\frac{1}{2}+\frac{1}{3}-\ldots$;
2. U is infinite, as in the case of $1+\frac{1}{2}+\frac{1}{3}+\ldots$;
3. U cannot be infinite, but is not determined, as in the case $1-1+1-\ldots$;
4. U can be infinite and is not determined, as in the case $1-2+2-3+3-\ldots$

The object of du Bois-Reymond's memoir of 1870 was to exhibit some new relations between the first and the third case.

Du Bois-Reymond[13] formulated the fundamental principle of convergence as follows: "U is finite and determined or U is not, according as the sum $u_m+u_{m+1}+\ldots+u_n$ has zero for its limit when m and n become infinite with any relative velocity." On this principle he remarked: It is easy to show that if U is finite and determined, $u_m+\ldots+u_n$ must vanish when both m and n become infinite. It is also easy to show that U cannot be finite and determined if $u_m+\ldots+u_n$ does not vanish for every relative velocity with which m and n become infinite. It is not so easy to prove that, if $u_m+\ldots+u_n$ always vanishes when m and n become infinite, U cannot become indefinite or infinite. Although the above principle is often used, I do not know a rigid proof of it, and here I will simply assume it and will give the proof, which does not seem to me capable of being carried through without the conception of the definite integral, elsewhere." Du Bois-Reymond, added a note to say that this proof would be given in a work, shortly to be published, on the conver-

[12] *Cf. Ostwald's Klassiker* No. 185, p. 3.

[13] *Ibid.*, p. 5.

gence and divergence of definite integrals and infinite series. This publication never took place [14].

The idea that the sufficiency of Cauchy's criterion can be proved by means of considerations of definite integrals rests on an error in point of principle.[15] In fact, the very proof of the existence of a definite integral presupposes the existence of a limit to a sequence which satisfies Cauchy's criterion. It is true that, in the present case, the question of principle of the existence of a limit is somewhat obscured by du Bois-Reymond's general consideration of series whose sum is indefinite between finite limits. This conception was mentioned by Cauchy in his *Cours* of 1821.[16] A simple example of a function which tends to no one definite limit when the real variable x tends to a certain point, but for which there are finite upper and lower limits, such that the values of the function never go outside these limits, is given by $\sin(1/x)$ as x tends to zero. Here what du Bois-Reymond called "the limits of indefiniteness" are $+1$ and -1. The conception in question seems, if we except Abel, not to have attracted much attention. It was rediscovered and applied widely by du Bois-Reymond in 1870, and again independently discovered and used by Jacques Hadamard in 1888; and just before that time, the importance of the conception in the theory of functions had been emphasized by Otto Stolz and Moritz Pasch." [17]

Du Bois-Reymond [18] proved the following theorem: "Whatever may be the nature of the series $u_1+u_2+\ldots$, there are always two numerical magnitudes B and A, of which the greater B, for infinite values of n, is not surpassed by the series $U_n=u_1+u_2+\ldots u_n$, but on the other hand, is infinitely often reached, while similarly, for n infinite, the sum U_n does not sink below A, but is infinitely often equal to A. Thus, all the values U of which limit $_{n=\infty} U_n$ is capable are included between the limits A and B, and these are the narrowest limits in which we can conclude the values U. We will call the magnitudes A and B the "limits of indefiniteness" (*Unbestimmtheitsgrenzen*) and the difference $B-A$ the "interval of indefiniteness" of the series $u_1+u_2+\ldots\ldots$

"Proof. Let us first prove the existence of the upper limit of indeterminateness B. For every n there must be a least magnitude B_n which is not surpassed by $U_{n+m}=u_1+u_2+\ldots+u_{n+m}$, when m increases from

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[14] *Ibid.*, pp. 98-99.

[15] Cf. the remark of Pringsheim's quoted on *ibid.*, p. 98. Du Bois-Reymond's tendency to this confusion in point of principle is also illustrated by pp. 6, 99 (note 4).

[16] *ibid.*, p. 99.

[17] Cf. *ibid.*, pp. 99-100, 102-103. [18] *Ibid.*, pp. 7-8.

0 to ∞, since otherwise U_{n+m} would have to increase, either only increasingly or alternately increasingly and decreasingly above every limit, which would be contrary to the supposition that U cannot become infinite. This magnitude B_n does not increase for any value of n, that is to say $B_n - B_{n+1}$ is never negative, because we have supposed B_n is the least value which is not surpassed by U_{n+m}. Since now B_n never increases with increasing n, B_n must either approach a finite definite limit or not. In the latter case, B_n can only approach negative infinity [19], but this is excluded by the supposition. Thus B_n approaches as n increases, a finite and determined limit, which we will call B. This limit must be reached infinitely often for $n = \infty$. For, if it does not reach this limit infinitely often, it would not be reached any more for a finite value N of n and onwards. Then

$$B = B_n = U_n = u_1 + u_2 + \dots + u_n,$$

and B_{N+1} would be smaller than B_n by a finite magnitude. But then B could never be equal to B_n, since B is the limit to which the least values B_n must approach as n increases to infinity. Therefore U is infinitely often equal to B. The second part of the proof concerning A is proved quite similarly so that I will pass it over."

In this proof there lies implicitly a reference to a previous definition of irrational numbers, and we saw, when mentioning Stolz's "proof" of the sufficiency of Cauchy's criterion by the help of these very conceptions. This fundamental error of du Bois-Reymond has been strongly emphasized by Pringsheim.[20]

For many years after 1821, Cauchy had, perhaps, an almost exclusive influence on the development of mathematical thought relating to the fundamental question on convergence, and he shares with nobody the glory of first considering the questions of the existence of a definite integral and of the solutions of differential equations. In these questions, convergent sequences of numbers play an essential part. Indeed, the arithmetical formulation of the conception of the definite integral of a continuous function which Cauchy gave in 1823 has been regarded [21] as the decisive turning point in the development of the idea that it is necessary to prove the existence of a limit of a series which is defined

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[19] "When a function of a variable, while this variable increases to infinity either nowhere decreases or nowhere increases, the function must either approach a finite and definite limit, or it becomes $+\infty$ in the first case and $-\infty$ in the second. It cannot become indefinite."

[20] Cf. *ibid.*, pp. 101-102.

[21] By Pringsheim, *Encykl. der math. Wiss.*, I. A. 3, p. 65. Cf. above.

arithmetically. But we must remember that, for Cauchy, what are called "real numbers" had a geometrical foundation. With him, in fact, a "number" is a result of measurement.[22]

VIII.

It was the avoidance of the logical error of defining an irrational number b as the sum Σa_v of an infinite series—for the definition of the sum Σa_v is first obtained by putting it equal to the necessarily already defined number b—that gave rise to the modern theories of irrational numbers. "I believe", said Cantor,[23] "that this logical error, which was first avoided by Weierstrass, was almost universally passed over in earlier times, and was not remarked because it is one of the rare cases in which actual errors do not give rise to any important errors in calculation. Nevertheless, I am convinced that with this error hang together all the difficulties which have been found in the conceptions of the irrational, and, when this error is avoided, the irrational number is established with the same definiteness and clearness in our minds as the rational number". And again,[24] the number b is not defined as the limit of the terms a_v of what Cantor called, in his theory of irrational numbers, a "fundamental series" (a_v), as this would be a similar logical error, because then the existence of a limit would be presumed. In short, we do not need to obtain an irrational number, in the first place, by a limit-process, but, on the contrary, we become convinced of the practicability and evidence of limiting-processes in general by means of our *possession* of it.

IX.

We will now glance at the introduction of limits and irrational numbers in some text-books.

De Morgan's works are especially interesting to the historian of mathematics as showing both the influence of Cauchy, that of the later methods of the older analysts, and that of De Morgan's own powerful and logical mind. Thus we have seen, in one of the preceding notes, that the conception of convergence is formulated after Cauchy, and we must add that the equation

$$-1=1+2+4+8+\dots$$

is justified because the series "is the result of an attempt to procure an arithmetical result, upon an arithmetical process, to represent a

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[22] Cf. §IV above.

[23] *Math. Ann.*, Vol. XXI, 1883, p. 566.

[24] *Ibid.*, pp. 568-569.

quantity which is not arithmetical."[25] In another paper,[26] I have shown that the conceptions of Cauchy and Euler of the "continuity" of a function, were given side by side. And, in the same work, the existence of limits was founded on the postulate which appeared with Peacock, Hankel, and others in the attempt to make algebra a formal science independent of geometry. De Morgan, we know,[27] was keenly conscious of the fact that the system of points on a line is richer, as Dedekind has expressed it, than the system or rational numbers; and he probably did not contemplate the possibility of a purely arithmetical theory of irrational numbers.

Dealing with the subject of *limits*, De Morgan[28] said: "In the questions which occur in arithmetic and algebra, relating to problems the conditions of which can be satisfied only as nearly as we pleased but not exactly, it is usual to create a solution by hypothesis, and to say that we continually approach to that solution, the more nearly we solve the problem. Thus it is never said that there is no such thing as x, which makes x^2 actually equal to 6; but it is said that there is such a thing as the square root of 6, and it is denoted by $\sqrt{6}$. But we do not say we actually find this, but that we *approximate* to it. This non-existing limit, if we may so call it, actually has a more definite existence in geometry than in arithmetic, but only when we make a sort of supposition which is practically as impossible as the extraction of the square root of 6 in arithmetic. Let there be such things as geometrical lines, namely, lengths which have no breadth or thickness, and let it be competent to us to mark off points which divide one part of a line from another, without themselves filling any portion of space; then it is shown in Euclid that the side of a square which contains six square units is a line, which, when we come to apply arithmetic to geometry, must be called $\sqrt{6}$ whenever our arbitrary linear unit is called 1. And the lines represented by the preceding twelve fractions will, in such case, be a set of lines which being always greater than the line in question, yet are severally nearer and nearer to it. This line can no more be expressed

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25 *The Differential and Integral Calculus*, London, 1842, p. 224.

26 "The origin of Cauchy's conceptions of a definite integral and of the continuity of a function", *Isis*, Vol. I, 1914, pp. 661-703.

27 Cf. his little book *On the Connexion of Number and Magnitude: an Attempt to Explain the Fifth Book of Euclid*, London, 1836.

28 *Calculus*, pp. 7-8.

by means of an arithmetical fraction than $\sqrt{6}$." A "limit" was defined[29] as a certain quantity 1, so that..." [30].

Arthur Rösler[31] examined sixteen text-books of dates varying from 1776 to 1881 with respect to the definition of irrational numbers, and found that, in essentials, they all came under the four heads:

"1. An irrational number is the ratio of two continuous incommensurable, homogeneous magnitudes.

"2. An irrational number is a number (numerical form, numerical magnitude), whose value can be given only approximately but up to any desired degree of approximation by means of rational numbers.

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[29] *Ibid.*, p. 9.

[30] In this connexion, an interesting study is afforded by G. Chrystal's ***Algebra***, *an Elementary Text-book for Higher Classes of Secondary Schools and for Colleges* Part ii., Edinburgh, 1889. On p. 103, Chrystal concluded that Cauchy's criterion of convergence was a sufficient condition for existence of a limit, because, "since S_n is finite for all values of n, the limit S_n cannot be infinite". It is not true that, if $f(x)$ is finite at each x, the upper limit of $f(x)$ is necessarily finite; nor—and it is this fact that concerns us here—is there the least reason for thinking that a series which has not a rational limit has a limit at all, unless the system of real numbers has been previously defined. This Chrystal did not do until his second edition was published. On p. 78 of the same volume, Chrystal, having proved that the values of $(1-1/x)^{-x}$ and $(1+1/x)^{x}$ cannot pass each other when x is increased, concluded: "Hence, when x is increased without limit, $(1-1/x)^{-x}$ must diminish down to a finite limit A, and $(1+1/x)^{x}$ must increase up to a finite limit B", and $A=B$.

In the second edition of this book, however, Chrystal attained to an exposition of the modern theories of irrational numbers.

Axel Harnack in his well-known text-book, though he professed to give an arithmetical account of irrational numbers, really fell back on geometry. On *An Introduction to the Study of the Elements of the Differential and Integral Calculus*, translated by G. L. Cathcart, London: 1891, p. 9, § 6, he says that a fundamental sequence" "defines a number". On p. 8 (§ 5) after determining two monotone sequences (α) and (β) of rational numbers, such that α^n and β^n are always coming nearer to the value a/b (which has no rational nth root), so that they ultimately differ from it as little as we please, Harnack proceeded; "Hence it appears that the numbers α and β themselves also approach more and more to one definite quantity, which, even though it does not exist among rational numbers, is yet called a numerical value, because it is connected with a rational number by a perfectly determinate arithmetical operation." This conclusion may possibly rest on geometrical grounds: it does not rest on logical grounds.

At the beginning of the book (p. 1), Harnack remarked: "To describe thoroughly the phenomena of motion is to assign every circumstance in numbers of concrete units: so that, if the series of numbers is also to enable us to describe motion, it must contain a continuous series of quantities."

[31] *Die neueren Definitionsformen der irrationalen Zahlen und ihre Bedeutung fur die Schule*, Hannover, 1886, pp. 6-7.

"3. An irrational number is a fraction whose numerator and denominator have infinitely many figures, or, what is the same thing, it is an infinite, non-periodic decimal fraction.

"4. An irrational number is the limit—not representable by means of rational numbers—of an infinite sequence of rational magnitudes, or it is the limit of a continually increasing and yet finite magnitude.

"The first and oldest of these definitions is based upon the contents of the tenth book of Euclid's *Elements*. This definition, though correct in itself, has the defect of resting on geometrical considerations, whereas we must wish that the conceptions of arithmetic be developed solely out of arithmetic. Also we must notice that the conception of the ratio between two incommensurable magnitudes can only be developed clearly and understood when the irrational numbers are already introduced.

"The second definition is logically false; it is too narrow a definition as long as we have not given such a development of the conception "number" that the conception of irrational number is contained in it...... How is a clear-thinking scholar to understand this definition, if he is not shown beforehand that, besides the rational numbers, there are yet other numbers just as definite?

"The uselessness of the third definition can be recognized at the first glance; we cannot define a conception which is, for us, to denote something definite by something indefinite unfinished, in a fluid state, a Being by a Becoming.[32]

"The fourth definition, like the second, presupposes that the conception of limit is already given in such a manner that it comprises the conception of irrational number. In the contrary case we have a mere tautology. This error, of tacitly postulating the conception of limit for certain infinite series before a criterion for the existence and a foundation for the idea of such a series-limit was given, was often passed over."

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[32] "An infinite non-periodic decimal fraction can only, I think, be taken to mean the idea (which naturally cannot be completed) of an infinite set of the rational numbers represented by the various finite decimal fractions written That there is a fixed number which is the least number greater than all these rational numbers cannot be concluded from the fact that, geometrically, it seems likely to some of us that there ought to be a point which would correspond to the limit in question."

X

We will now shortly notice a few discussions of irrational numbers which were published before 1872 the year in which the three principal modern theories became known to the public at large by publication.

In 1847 Guilmin[35] attempted to give arithmetical definitions of certain irrational numbers ; and, in the course of his paper he said : [34]

"The roots approached from below [approximations to $\sqrt{2}$] form an increasing sequence ; however, these numbers have an upper limit, for some of them can surpass any one of the corresponding roots from above."

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[35] "Définitions précise arithmetiques des racines et des logarithmes incommensurables", *Nouv. Ann*, Vol VI, 1847, 318-326,. On p. 395 is an announcement of a *Cours complet d' arithmétique... ..*, by A. Guilmin, Paris, 1847. On p. 313 of his paper, he refers to p. 110 of the same volume. Here is part of an anonymous (by O. Terquem, see the index, pp. 491, 499) paper "Question d'examen. Theorie des exposants de nature quelconque (v. t. V., p. 704)", on pp. 106-113. The part dealing with real irrational exponentials on pp. 109-110 (errata on p. 504) ;

"*Definition.*—The symbol m being a whole number and p being a number which is not a power of index m, the symbol $m\sqrt{p}$ denotes that there exists an infinite series of *finite* numbers, such that by raising them all to the mth power, we obtain a second series which has p for limit ; *i.e.*, a second series of which no term$=p$, but where the difference from above or from below (*en exces ou en défaut*), between the terms and p can become $<$ any given quantity. In the last series the difference between two consecutive terms can also descend below every given quantity ; but it has not a limit assignable by a *finite* number of figures, in any system of numeration ; while the second series has assignable limit. Take, for example, $\sqrt{2}$, we have for first series 1, 3/2, 7/5, 17/12, 19/41,... the differences between the terms continually diminish ; it has not a limit expressible in figures of a numeration. We denote symbolically this limit by $\sqrt{2}$ *i.e.*, by forming the second series 1, 9/4, 49/25, 289/144, etc., the limit 2.

"*General observation.*—Whenever we perform any one of the six arithmetical operations on irrational expressions, we must always understand, if we are to know what we are saying, that we perform these operations on the rational expressions of the first series of which these irrational expressions represent the symbolic limit".

[34] *Ibid.*, p. 316.

In Joseph Bertrand's Traité d'arithmètique[25] occurs the passage :

"... We suppose that this definition [of an incommensurable number] consists in indicating what are the commensurable numbers which are smaller or greater than itself."

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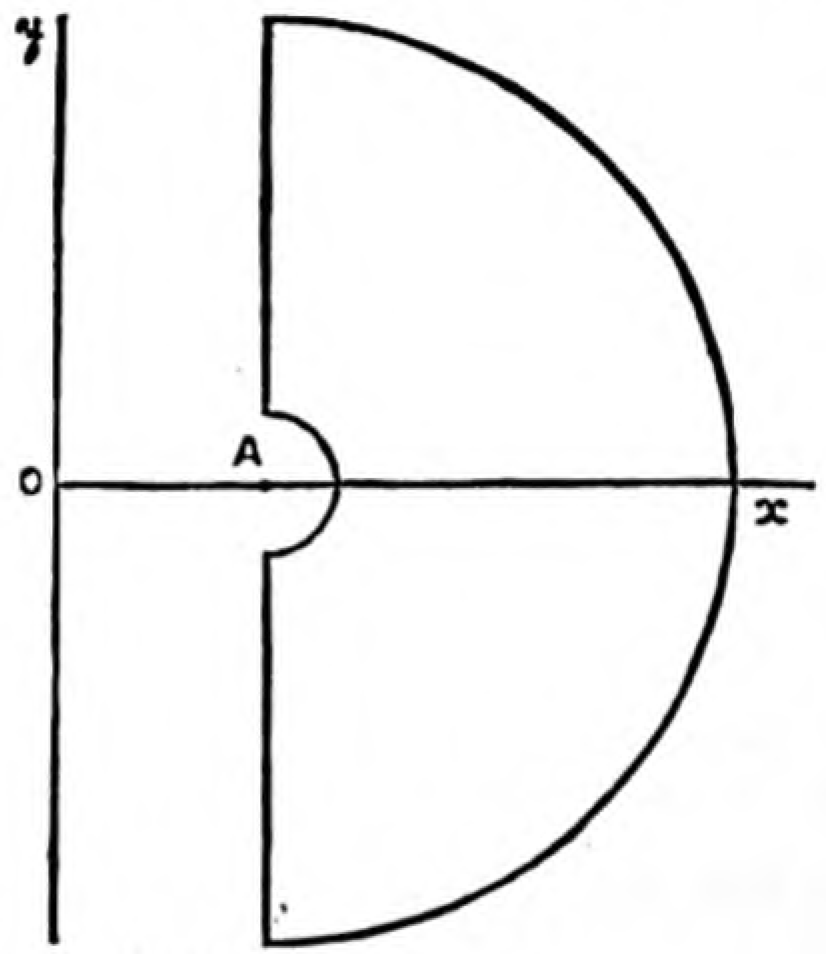
[25] Paris, 1851, p. 231; 12th ed., Paris, 1901, p 271. On p. 233 of the 12th ed. Bertrand defined "the magnitude of which $\sqrt{N}$ [N being the square of no whole or fractionary number] is the measure", and said that "it is not possible to define directly an abstract number. If one reflects on the definitions given, given in the simple cases of whole or fractionary numbers, one will see that they are only the indication of the operation by the aid of which the magnitude of which they are the measure is derived from the unit". The same is said for a cube root on p. 255. And on p. 271: "An incommensurable number can only be defined by indicating how the magnitude which it expresses can be formed by means of the unity. In what follows, we suppose that one can then conceive the magnitude of which it is the measure, as serving as common limit to those which are represented by greater or smaller numbers". "To add or to subtract two incommensurable numbers is to find a number expressing the sum or the difference of the magnitude expressed by the proposed numbers". (*Ibid*).

SHORT NOTES.

On Question 494.

Prove that $\int_0^\infty \frac{\phi(a+ix)-\phi(a-ix)}{2ix}dx = \frac{\pi}{2}\Big\{\phi(\infty)-\phi(a)\Big\}$.

1. The solution given in the I. M. J. of April 1914, p. 65 is not rigorous and the sign of the right hand side is incorrect. Very elementary considerations of the theory of functions afford both a rigorous proof and the conditions under which the theorem is valid.



2. Consider $\int_C \frac{\phi(z)}{z-a}dz$ round a contour consisting of two semi-circles centre (a, o) and radii r, R where $z \to o$, $R \to \infty$ and bounded by the broken part of the diameter whose equation is $x = a$.

We have

$$\int_r^R \frac{\phi(a+iy)}{iy}idy - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\phi(a+Re^{i\theta})}{Re^{i\theta}}iRe^{i\theta}d\theta$$

$$-\int_r^R \frac{\phi(a-iy)}{iy}idy + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\phi(a+re^{i\theta})}{re^{i\theta}}ire^{i\theta}d\theta = 0$$

provided $\phi(z)$ is nowhere infinite within or on the contour of integration.

This becomes

$$\int_0^\infty \frac{\phi(a+ix)-\phi(a-ix)}{2ix}dx = \frac{\pi}{2}\Big\{\phi(\infty)-\phi(a)\Big\}.$$

The conditions to be satisfied by $\phi(z)$ are that $\phi(\infty)$ is finite and that $\phi(z)$ has no infinities to the right of $z=a$.

Example 1. $\phi(z)=e^{-z}, a=0, \int_0^\infty \frac{\sin x}{x} dx=\frac{\pi}{2}$.

Example 2. If $\phi(z)$ has infinities within the contour, they must be surrounded by circles of infinitesimal radius; as $\phi(z)=\frac{1}{z-b}$, $b>a$, gives

$$\int_0^\infty \frac{\frac{1}{a-b+ix}-\frac{1}{a-b-ix}}{x} dx+\int_{(b)} \frac{1}{(z-b)(z-a)} dz$$

$$=i\pi\left\{-\frac{1}{a-b}\right\} \text{ since } \phi(\infty)=0.$$

$$\therefore \qquad -2i\int_0^\infty \frac{dx}{(a-b)^2+x^2}+\frac{2i\pi}{b-a}=\frac{i\pi}{b-a},$$

or

$$\int_0^\infty \frac{dx}{(a-b)^2+x^2}=\frac{\pi}{2(b-a)}, \text{ for } b>a.$$

3. Consider $\int \frac{1-e^{-2nz}}{\sinh z}\frac{dz}{z-a}$ taken over a contour bounded by the axis of y, an infinite semicircle, centre the origin, on the positive side of the axis of x, an infinitesimal circle centre (a, o), when a is supposed positive. There are no infinities within the contour; for $\sinh z=0$ gives $e^{2z}=1=e^{2ir\pi}$, but for all these values $\frac{1-e^{-2nz}}{1-e^{-2z}}$ is finite when x is a positive integer.

$$\therefore \int_0^\infty \frac{1-e^{-2nix}}{i\sin x}\frac{1}{ix-a}\, idx-\int_0^\infty \frac{1-e^{2nix}}{-i\sin x}\cdot\frac{1}{-ix-a}\,(-i)dx$$

$$+2\pi i\frac{1-e^{-2na}}{\sinh a}=0.$$

Simplifying we get

$$\int_0^\infty \frac{a\sin 2nx+x(1-\cos 2nx)}{x^2+a^2}\frac{dx}{\sin x}=\pi\frac{1-e^{-2na}}{\sinh a}.$$

Again consider $\int \frac{1-e^{-2nz}}{\sinh z}\frac{dz}{z+a}$ over the contour bounded by the axis of y and the infinite semicircle, centre the origin, on the positive side of the axis of x, a being positive.

$$\int_0^\infty \frac{x(1-\cos 2nx)-a\sin 2nx}{x^2+a^2}\frac{dx}{\sin x}=0.$$

Hence

$$\int_0^\infty \frac{\sin 2nx}{x^2+a^2}\frac{dx}{\sin x}=\frac{\pi}{a}e^{-na}\frac{\sinh na}{\sinh a}$$

and

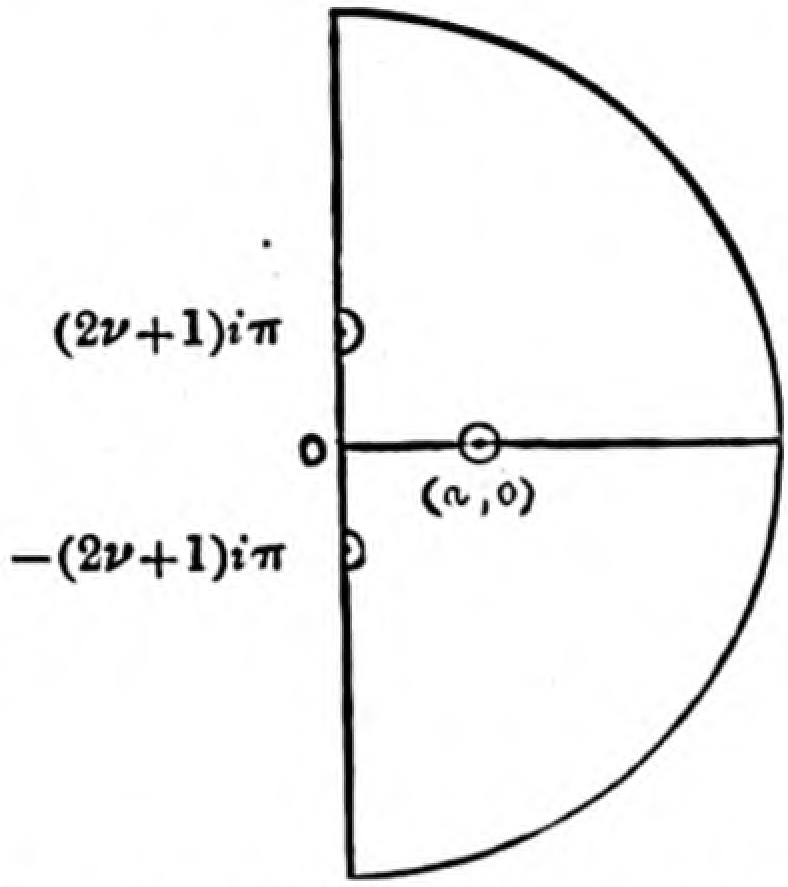
$$\int_0^\infty \frac{x\sin^2 nx}{x^2+a^2}\frac{dx}{\sin x}=\frac{\pi}{2}e^{-na}\frac{\sinh na}{\sinh a}.$$

Here n is necessarily a positive integer as $\phi(\infty)$ was assumed to be zero.

4. The special case $2n=-1$ can also be investigated as follows:

Consider
$$\int \frac{1}{1+e^{-z}}\,\frac{dz}{z-a}.$$

The function $1+e^{-z}$ is zero for $z=(2r+1)\pi i$. Surround these points by semicircles as in figure 2 and integrate over the contour thus obtained.



For the point $z=(2r+1)i\pi$, if we take the integral
$$\int_{-r}^{+r} \frac{1}{1-e^{-iu}}\,\frac{iu}{iu+(2r+1)i\pi-a}=0$$

i. e.
$$\int_{-r}^{+r} \frac{du}{u}=0,$$

we can write
$$\int_0^\infty \frac{1}{1+e^{-ix}}\,\frac{i\,dx}{ix-a}-\int_0^\infty \frac{1}{1+e^{ix}}\,\frac{i\,dx}{ix+a}-\pi\phi(\infty)+2\pi i\,\frac{1}{1+e^{-a}}$$
$$+i\pi\sum_0^\infty -\frac{2a}{(2r+1)^2\pi^2+a^2}=0.$$

For, the integral round the semicircle at $(2r+1)i\pi$ is, by putting $z=(2r+1)i\pi+re^{i\theta}$, equal to
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{1-e^{-re^{i\theta}}}\,\frac{re^{i\theta}\,i\,d\theta}{(2r+1)\pi i-a}=\frac{i\pi}{(2r+1)i\pi-a}.$$

Hence for the point $\pm(2r+1)i\pi$, we get $\dfrac{-2i\pi a}{(2r+1)^2\pi^2+a^2}$.

Also $\phi(\infty)=1$.

Hence

$$\int_0^\infty \frac{x \sin \frac{x}{2} - a \cos \frac{x}{2}}{(x^2+a^2)\cos\frac{x}{2}} dx = \pi - \frac{2\pi}{1+e^{-a}} + \sum_0^\infty \frac{2\pi a}{(2r+1)^2\pi^2+a^2}$$

$$= \pi - \frac{2\pi}{1+e^{-a}} + \pi \frac{1-e^{-a}}{1+e^{-a}}.$$

Writing $-a$ for a, and omitting the contour integral round (a, o), we have also

$$\int_0^\infty \frac{x \sin \frac{x}{2} + a \cos \frac{x}{2}}{(x^2+a^2)\cos\frac{x}{2}} dx = \pi - \pi \frac{1-e^{-a}}{1+e^{-a}}.$$

Subtracting we have $\displaystyle\int_0^\infty \frac{dx}{x^2+a^2} = \frac{\pi}{2a}$;

and adding we have $\displaystyle\int_0^\infty \frac{x \tan\frac{1}{2}x}{x^2+a^2} dx = \frac{\pi}{1+e^{a}}$,

where the principal value of the integral is determined.

5. In the formula of § 2, put $\phi(z) = \dfrac{e^{-r(z-a)}}{\sinh z}$; thus $\phi(\infty) = 0$.

$$\therefore \quad \int_0^\infty \left\{ \frac{e^{-rix}}{\sinh(a+ix)} - \frac{e^{rix}}{\sinh(a-ix)} \right\} \frac{dx}{ix} = \pi \left(-\frac{1}{\sinh a} \right)$$

This simplifies to

$$\int_0^\infty \frac{e^a \sin(r+1)x - e^{-a} \sin(r-1)x}{\cosh 2a - \cos 2x} \frac{dx}{x} = \frac{\pi}{2 \sinh a}.$$

Put $r=0$, $\displaystyle\int \frac{\sin x}{\cosh 2a - \cos 2x} \frac{dx}{x} = \frac{\pi}{2 \sinh a}, \frac{1}{e^a+e^{-a}}.$

$$\text{and } \int_0^\infty \frac{e^a \sin(2r+1)x}{\cosh 2a - \cos 2x} \frac{dx}{x} = \frac{\pi}{2\sinh a} + e^{-2a} \int_0^\infty \frac{e^a \sin(2r-1)x}{\cosh 2a - \cos 2x} \frac{dx}{x}$$

$$= \frac{\pi}{2 \sinh a} \left\{ 1 + e^{-2a} + e^{-4a} + - + e^{-2(r-1)a} + e^{-2ra} e \frac{e^a}{e^a+e^{-a}} \right.$$

$$= \frac{\pi}{2 \sinh a} \left\{ \frac{1-e^{-2ra}}{1-e^{-2a}} + \frac{e^{-2ra}}{1+e^{-2a}} \right\} = \frac{\pi}{2 \sinh a} \left\{ \frac{1}{1-e^{-2a}} - \frac{2e^{-2(r+1)a}}{1-e^{-4a}} \right\}.$$

$$\therefore \int_0^\infty \frac{\sin(2r+1)x}{\cosh 2a - \cos 2x} \frac{dx}{x} = \frac{\pi}{2 \sinh a} \left\{ \frac{\pi}{2 \sinh a} - \frac{e^{-(2r+1)a}}{\sinh 2a} \right\}.$$

Put $r=1$, and

$$\int_0^\infty \frac{\sin 2x}{\cosh 2a - \cos 2x} \frac{dx}{x} = \frac{\pi e^{-a}}{2 \sinh a},$$

and $$\int_0^\infty \frac{e^a \sin 2rx}{\cosh 2a - \cos 2x} \frac{dx}{x} = \frac{\pi}{2 \sinh a} \left\{ 1 + e^{-2a} + \ldots + e^{-2(r-1)a} \right\}$$

$$\therefore \qquad \int_0^\infty \frac{\sin 2rx}{\cosh 2a - \cos 2x} \frac{dx}{x} = \frac{\pi(1-e^{-2ra})}{4 \sinh^2 a}.$$

6. Similarly, applying the same method to $\phi(z) = \dfrac{e^{-r(z-a)}}{\cosh z}$, we may obtain

$$\int_0^\infty \frac{e^a \sin (r+1)x + e^{-a} \sin (r-1)x}{\cosh 2a + \cos 2x} \frac{dx}{x} = \frac{\pi}{2 \cosh a};$$

whence $$\int_0^\infty \frac{\sin (2r+1)x}{\cosh 2a + \cos 2x} \frac{dx}{x} = \frac{\pi}{2 \cosh a} \left\{ \frac{1}{2 \cosh a} + (-)^r \frac{e^{-(2r+1)a}}{\sinh 2a} \right\},$$

and $$\int_0^\infty \frac{\sin 2rx}{\cosh 2a + \cos 2x} \frac{dx}{x} = \frac{\pi(1-(-)^r e^{-2ra})}{4 \cosh^2 a}.$$

A. C. L. WILKINSON.

Note on the length of a Circular Arc.

1. Let s denote the arc of a circle of unit radius subtending an angle θ at the centre. Then $s=\theta$. Assume that

$$s = \theta = a_1 \sin p\theta + a_2 \sin p^3\theta + a_3 \sin p^5\theta + \ldots a_n \sin p^n\theta, \qquad \ldots \quad (1)$$

where $p<1$ and $a_1, a_2 \ldots a_n$ are unknown. We shall find values for $a_1, a_2, \ldots a_n$ in terms of p, so that equation (1) may hold good approximately.

Expanding the sines on the right side and neglecting terms beyond θ^{2n-1}, we have the following n equations for $a_1, a_2, \ldots a_n$:

$$\left.\begin{array}{lllll} a_1p + a_2p^2 + & \ldots & \ldots & a_np^n = 1 \\ a_1p^3 + a_2p^6 + & \ldots & \ldots & a_np^{3n} = 0, \\ a_1p^5 + a_2p^{10} + & \ldots & \ldots & a_np^{5n} = 0, \\ \ldots \quad \ldots & \ldots & \ldots & \ldots \\ \ldots \quad \ldots & \ldots & \ldots & \ldots \\ a_1p^{2n-1} + a_2p^{4n-2} + & \ldots & \ldots & a_np^{(2n-1)} = 0. \end{array}\right\} \qquad \ldots \quad (2)$$

Now, consider the equation

$$\phi(x) \equiv a_nx^{n-1} + a_{n-1}x^{n-2} + \ldots\ldots a_1 = 0. \qquad \ldots \qquad \ldots \quad (3)$$

Its roots are $p^3, p^5, \ldots\ldots p^{2n-1}$, by virtue of (2), and we may write therefore

$$\phi(x) \equiv a_n(x-p^3)(x-p^5)\ldots(x-p^{2n-1})$$
$$\equiv a_n \{ x^{n-1} - P_1x^{n-2} + P_2x^{n-3} - \ldots\ldots(-1)^{n-1} P_{n-1} \} \quad \ldots \quad (4)$$

Comparing (3) and (4), we deduce that

$$a_n = \frac{a_{n-1}}{-P_1} = \frac{a_{n-2}}{P_2} = \ldots \frac{a_1}{(-1)^{n-1} P_{n-1}}.$$

Also from the first relation in (2)

$$p\phi(p)=1.$$

$$\therefore \qquad a_n=\frac{1}{p^n(1-p^2)(1-p^4)\ldots(1-p^{2n-2})} \qquad \text{by (4).}$$

Hence

$$a_r=\frac{(-1)^{n-r}\,\mathrm{P}_{n-r}}{p^n(1-p^2)\,(1-p^4)\ldots\ldots(1-p^{2n-2})}.$$

2. The error in neglecting powers of θ beyond θ^{2n-1} in the expansion of $\sin\theta$ is $(-1)^{n+1}\,\mathrm{R}_n$, where $\mathrm{R}_n<\dfrac{\theta^{2n+1}}{(2n+1)!}$. Hence a maximum limit to the error in equation (1) is

$$\frac{(-1)^{n+1}\,\theta^{2n+1}}{(2n+1)!}\left\{a_1\,p^{2n+1}+a_2\,p^{4n+2}+\ldots\ldots a_n\,p^{n(2n+1)}\right\}$$

$$=\frac{(-1)^{n+1}\,\theta^{2n+1}}{(2n+1)!}\;\frac{p^{2n+1}\phi(p^{2n+1})}{p\;\phi\;(p)}=\frac{\theta^{2n+1}}{(2n+1)!}p^{n(n+1)}$$

3. *Examples:*

(1) Put $p=\frac{1}{2}$, $n=2$; we have $a_1=-\frac{2}{3}$, $a_2=\frac{16}{3}$.

$\therefore \quad s=\theta=\frac{16}{3}\sin\frac{1}{4}\,\theta-\frac{2}{3}\sin\frac{1}{2}\,\theta.$

$=\frac{1}{3}(8\mathrm{H}-\mathrm{C}),$ [Huyghen's]

where C is the chord of the whole arc, and H is the chord of half the arc.

(2) If $p=\frac{1}{2}$, $n=3$, we find $a_1=\frac{2}{45}$, $a_2=-\frac{16}{9}$, $a_3=\frac{512}{45}$, so that

$$s=\theta=\tfrac{512}{45}\sin\tfrac{1}{8}\theta-\tfrac{16}{9}\sin\tfrac{1}{4}\theta+\tfrac{2}{45}\sin\tfrac{1}{2}\theta$$

$$=\frac{256\mathrm{Q}-40\mathrm{H}+\mathrm{C}}{45}.$$ [Edward's *Diff. Calc.*, p. 109, Ex. 23(2).]

where Q is the length of quarter the arc.

R. VYTHYNATHASWAMY.

Geometrical Exercises with the Straight Edge.

The straight edge or ruler, as a mathematical instrument, enables as to draw straight lines of indefinite length through one or more points. It is not helpful to set off definite lengths. Geometrical constructions in which the straight edge only is to be used naturally find a place in Projective Geometry, (Geometrie der Lage, Geometrie de Position) in which metrical considerations are not, as in Euclidean Geometry, of primary importance. See : J. W. Russell's *Elementary Treatise on Pure Geometry.*

An attempt is here made to introduce constructions with the straight edge at an earlier stage in connection with Euclid's *Elements.* The fundamental propositions on which the scheme is based are two.

Prop. I. In any triangle, the diagonals of the trapezoid formed by drawing a parallel to one of the sides intersect on the median which bisects that side.

And conversely, if in a triangle, lines be drawn through any point on a median and the extremities of the side to which it is drawn, the line joining the meet of these lines with the other sides is parallel to that side.

Prop. II. If in a triangle OAA′, a line OB be drawn meeting AA′ in B and through any point P on OB, APC, A′PD be drawn meeting OA′ in C and OA in D, then DC meets AA′ in a fixed point B′ which cuts AA′ externally; in other words, B′ is so situated that AB : BA′ : : AB′ : B′A′.

Constructions with the straight edge can be made by paper folding, but the latter also admits of metrical constructions.

Constructions based on Prop. I can be effected with the parallel ruler.

Exercises.

1. Given a finite straight line and its mid-point, draw a line parallel to it through a given point.

2. Given two parallel segments, draw the line which bisects each of them.

3. Three segments and their mid-points being given describe triangles whose sides are parallel to these segments and pass through three given points.

4. Bisect a given straight line, being given,

(1) two lines which are not in the same direction and their mid-points;

(2) a parallelogram;

(3) a circle with its centre;

(4) three equidistant parallel lines.

5. Given two parallel segments produce one of them in increments each equal to its half, and divide the other into any number of equal parts.

6. Construct the join of a given point to the inaccessible point of intersection of two given straight lines.

7. Construct lines which shall be concurrent with a given line and the line joining two given points when this last line cannot be drawn.

8. Through a given point O draw a line cutting the sides BC, CA, AB of a triangle ABC in A′, B′, C′ such that (OA′, B′C′) may be harmonic.

9. Given a parallelogram and a circle find the centre of the circle.

10. Given a circle and its centre.

(1) Inscribe a rectangle in the circle.

(2) Draw the diameters parallel to the sides of the rectangle.

(3) Bisect a chord and its arc.

(4) Bisect an angle whose arms intersect the circle.

(5) Draw from a given point a perpendicular to a given line which cuts the circle.

(6) Draw the diameter which is perpendicular to a given line which does not cut the circle.

(7) Draw through a given point a perpendicular to a given diameter of the circle.

(8) Inscribe

(*a*) an equilateral triangle,

(*b*) a regular pentagon.

11. Given a circle with its centre and an inscribed triangle find the following points:—(*a*) centroid (*b*) in-centre (*c*) points of contact of the in-circle with the sides of the triangle (*d*) orthocentre (*e*) centre of nine-points circle (*f*) point of contact of the in-circle and nine-points circle (*g*) symmedian point (*h*) Brocard points (*i*) Jerabek points.

T. Sundara Row.

The Face of the Sky for January and February 1915.

Sidereal time at 8 p. m.

		December 1914.				*January* 1915.				*February* 1915.		
		H.	M.	S.		H.	M.	S.		H.	M.	S.
1	...	0	38	24	...	2	40	37	...	4	42	50
8	...	1	5	59	...	3	8	12	...	5	10	26
15	...	1	33	35	...	3	35	48	...	5	38	12
22	...	2	1	11	...	4	3	24	...	6	5	38
29	...	2	28	47	...	4	31	0				

From this table the constellations visible during the evenings can be ascertained by a reference to their positions as given in the star-charts.

Phases of the Moon.

		January.				*February.*		
		D.	H.	M.		D.	H.	M.
Full Moon	...	1	5	51 P.M.				
Last Quarter	...	9	2	43 A.M.	...	7	10	41 A.M.
New Moon	...	15	8	12 P.M.	...	14	10	1 „
First Quarter	...	23	11	2 A.M.	...	22	8	28 „
Full moon	...	31	10	11 „				

Eclipses.

On February 14, there will be an annular solar eclipse, invisible in India.

The Planets.

Mercury is in superior conjunction on January 5 ; and attains its greatest elongation (18° 14′ E) on February 6. It is stationary on February 12. It is in conjunction with the moon on January 16 and on February 15, with Mars on January 2, with Jupiter on February 2 and with Uranus on January 21.

Venus is a morning star. It attains its greatest brilliancy on January 2 and its greatest elongation (46° 54′ W) on February 6. It is in conjunction with the moon on January 12 and February 10. It is in Sagittarius in these months.

Mars is a morning star. It is in Sagittarius in January and near the boundary between Capricornus and Aquarius in February. It is in conjunction with the moon on January 5 and February 13 and with Uranus on February 15.

Jupiter is in conjunction with the sun on February 24. It is in conjunction with the moon on January 18 and February 15. It is near the boundary between Capricornus and Aquarius in January and in Pisces in February.

Saturn is stationary on February 26. It is in conjunction with the Moon on January 27 and February 24. Throughout these months it is in Taurus.

Uranus is in conjunction with the sun on February 1 and with the moon on January 16.

Neptune is in conjunction with the moon on January 3, 30 and February 26.

V. Ramesam.

SOLUTIONS.

Question 340.

(M. Bhimasena Rao) :—DEF is the pedal triangle of P with respect to a triangle ABC. If AD, BE, CF are concurrent at Q, show that PQ passes through the circumcentre of DEF and through a fixed point

Solution by N. Sankara Aiyar, M. A.

It is easy to see that if P be (f, g, h) then D is $0, g+f\cos C, h+f\cos B$; E is $f+g\cos C, 0, h+g\cos A$; and F is $f+h\cos B, g+h\cos A, 0$.

Hence AD, BE, CF are concurrent if

$$(g+f\cos C)(h+g\cos A)(f+h\cos B)=(g+h\cos A)\times(h+f\cos B)(f+g\cos C) \quad \ldots \quad (1)$$

On reduction this becomes

$$\Sigma(f^2g\cos B+fg^2\cos B\cos C)=\Sigma(f^2g\cos A\cos C+fg^2\cos A)$$

i.e., $$\Sigma f(g^2-h^2)(\cos A-\cos B\cos C)=0.$$

This shows that the line PP′, where P′ is the isog. conj. of P, passes through the fixed point $(\cos A-\cos B\cos C)$, etc.

But PP′ also passes through the circumcentre of DEF.

If AD, BE, PP′ are concurrent, it follows that Q lies on PP′ *i.e.*, PQ passes through the fixed point noted above and through the circumcentre of DEF.

Now AD is $\beta(h+f\cos B)-\gamma(g+f\cos C)=0$

BE is $\alpha(h+g\cos A)-\gamma(f+g\cos C)=0$

and PP′ is the line through P and $\cos A-\cos B\cos C$, &c. &c.; *i.e.*, PP′ is

$$\Sigma\alpha(g\,\overline{\cos C-\cos A\cos B}-h\,\overline{\cos B-\cos A\cos C})=0.$$

These three lines are concurrent if

$$(g\,\overline{\cos C-\cos A\cos B}-h\,\overline{\cos B-\cos A\cos C})(h+f\cos B)(f+g\cos C)$$

$$+(h\,\overline{\cos A-\cos B\cos C}-f\,\overline{\cos C-\cos A\cos B})(g+f\cos C)(h+g\cos A)$$

$$+(f\,\overline{\cos B-\cos A\cos C}-g\,\overline{\cos A-\cos B\cos C})(h+f\cos B)(h+g\cos A)=0.$$

Simplifying and writing l, m, n; l', m', n' for $g+h\cos A$, $h+f\cos B$, $f+g\cos C$, $h+g\cos A$, $f+h\cos B$, $g+f\cos C$, we get

$$(l\cos C-l'\cos B)\,mn+(m\cos A-m'\cos C)\,n'l'+(n\cos B-n'\cos A)\,ml'=0;$$

i.e. $lmn\cos C - l'mn\cos B + l'n'm\cos A - l'm'n'\cos C$

$+ l'mn\cos B - l'mn'\cos A = 0$;

i.e. $lmn\cos C - l'm'n'\cos C = 0.$,

which is true, since $lmn = l'm'n'$.

Hence AD, BE, CF are concurrent on PP′.

Question 441.

(S. Ramanujan) :—Shew that

$$(6a^2-4ab+4b^2)^3=(3a^2+5ab-5b^2)^3+(4a^2-4ab+6b^2)^3+(5a^2-5ab-3b^2)^3$$

and find other quadratic expressions satisfying similar relations.

Solution by S. Narayanan, B.A., L.T.

Let $(la^2-nab+nb^2)^3 \equiv (pa^2+mab-mb^2)^3+(na^2-nab+lb^2)^3$
$+(ma^2-mab-pb^2)$.

It is proposed to find the most general values of l, m, n, p. The conditions for the identity are seen to be

$$l^3=m^3+n^3+p^3 \quad \dots \quad \dots \quad \dots \quad (1)$$

and $$n(l^2-n^2)=m(m^2-p^2). \quad \dots \quad \dots \quad (2)$$

Writing (1) and (2) in the forms

$$l^3-n^3=m^3+p^3$$

$$n(l^2-n^2)=m(m^2-p^2),$$

and dividing, we have

$$\frac{l^2+ln+n^2}{n(l+n)}=\frac{m^2-mp+p^2}{m(m-p)};$$

i.e. $$1+\frac{l^2}{n(l+n)}=1+\frac{p^2}{m(m-p)}.$$

$$\therefore \quad \frac{l^2}{n(l+n)}=\frac{p^2}{m(m-p)},$$

or $$\frac{n(l+n)}{l^2}=\frac{m(m-p)}{p^2};$$

i.e. $$\left(\frac{n}{l}+\frac{1}{2}\right)^2=\left(\frac{m}{p}-\frac{1}{2}\right)^2,$$

i.e. $$\frac{n}{l}+\frac{1}{2}=\pm\left(\frac{m}{p}-\frac{1}{2}\right).$$

Hence $$\frac{n}{l}=\frac{m}{p}-1 \quad \dots \quad \dots \quad \dots \quad (3)$$

or $$\frac{n}{l}=-\frac{m}{p}. \quad \dots \quad \dots \quad \dots \quad (4)$$

Putting $l=\lambda p$ in (3), we get $n=\lambda(m-p)$.

Substituting in (1) for l and n

$$\lambda^3 p^3 = m^3 + \lambda^3 (m-p)^3 + p^3.$$

$\therefore$ $$\lambda^3 \{ p^3 - (m-p)^3 \} = m^3 + p^3.$$

i.e. $$\lambda^3 (2p-m)(p^2-pm+m^2) = m^3 + p^3$$

$\therefore$ $$\lambda^3 (2p-m) = m+p, \text{ since } p^2-pm+m^2 \neq 0.$$

$\therefore$ $$m(1+\lambda^3) = p(2\lambda^3-1).$$

$\therefore$ $$n(1+\lambda^3) = \lambda(m-p) = \lambda(\lambda^3-2)p.$$

Hence we may write

$$l = \lambda(\lambda^3+1),\ m = (2\lambda^3-1),$$
$$n = \lambda(\lambda^3-2),\ p = (\lambda^3+1),$$

which are the general solutions resulting from equation (3).

Again, from (4), we get

$$l = \lambda p,\ n = -m\lambda$$

$\therefore$ $$p^3\lambda^3 = m^3 - m^3\lambda^3 + p^3, \text{ substituting in (1).}$$

$\therefore$ $$(p^3+m^3)(\lambda^3-1) = 0.$$

Hence, either $p=-m$, and therefore $l=n$; or $\lambda=1$ and therefore $l=p$, $n=-m$; which are special solutions.

Question 512.

(A. Narasinga Rao) :—If $n\,\phi(n)+(n+2)\,\phi(n+2)=\phi(n+1)$, prove that

(i) $\phi(1)\,x - \phi(3)\,x^3 + \phi(5)\,x^5 \ldots = \phi(1)\sin(\tanh^{-1}x)$

(ii) $\phi(0) - \phi(2)\,x^2 + \phi(4)\,x^4 \ldots = \phi(0) + \phi(1)\{\cos\tanh^{-1}x - 1\}$.

Discuss the convergency of the series $\Sigma\phi(n)\,x^n$.

Additional Solution by the Proposer.

Multiply both sides of the given functional relation by x^{n+1}. Then

$$\frac{1}{x}.(n+2)\,f(n+2)\,x^{n+2} + x\,n\,f(n)\,x^n = f(n+1)x^{n+1}$$

$$\frac{1}{x}(n+1)\,f(n+1)\,x^{n+1} + x\,(n-1)\,f(n-1)\,x^{n-1} = f(n)\,x^n.$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\frac{1}{x}\,2\,f(2)\,x^2 + x.0.\,f(0) = f(1).x.$$

By addition

$$\left(x+\frac{1}{x}\right)\{2\,f(2)x^2 + 3\,f(3)x^3 + 4\,f(4)x^4 + \ldots \text{to } \infty\} + x^2 f(1) = \sum_1^\infty f(r).x^r.$$

Hence if $$u=\sum_1^{\infty} f(r)x^r,$$

we have $$(1+x^2)\frac{du}{dx}=u+f(1).$$

$\therefore$ $$u=Ce^{\tan^{-1}x}-f(1).$$

When $x=0$, $u=0$. $\therefore$ $C=f(1)$.

$\therefore$ $$\sum_0^{\infty} f(r)x^r=f(0)+f(1)\left\{e^{\tan^{-1}x}-1\right\}.$$

On replacing x by ix and equating real and imaginary parts on both sides, we get the two results given.

The series for $\tan^{-1}x$ is convergent or divergent according as
$$(x)\leq \text{ or } >1.$$

Also the series for e^x is convergent for all values of x.

An interesting point about f is its behaviour at infinity. Since $\Sigma f(n)$ is convergent $f(\infty)=0$, for otherwise the series would diverge or oscillate.

Also the test ratio of the series is $\text{Lt.}\,\dfrac{f(n+1)}{f(n)}=\text{Lt.}(t_n) \not> 1$. (1)

Our functional equation may now be written in the form
$$\frac{n+2}{n}\frac{f(n+2)}{f(n)}+1=\frac{f(n+1)}{f(n)}\frac{1}{n};$$

i. e. $$\underset{n\to\infty}{\text{Lt.}}\ (t_nt_{n+1}+1)=\text{Lt}\left(\frac{tn}{n}\right)=0. \quad \ldots \quad \ldots \quad (2)$$

If $t_n=t_{n+1}$ then each is equal to $\pm i$ from (2). This is impossible since all terms of the series are real.

Also $t_n\neq 0$ for then t_{n+1} is infinity which cannot be because of (1).

$\therefore$ $$t_n=-t_{n+1}=1.$$

Hence after a certain term the series consists of two positive and two negative terms.

Question 517.

(T. P. Trivedi, M.A., LL.B.):—Is there any known method of integrating completely
$$\left(\frac{d^2y}{dx^2}\right)^2\frac{d^5y}{dx^5}-\frac{d^2y}{dx^2}\frac{d^3y}{dx^3}\frac{d^4y}{dx^4}+\frac{40}{9}\left(\frac{d^3y}{dx^3}\right)^3=0,$$
which is the well-known differential equation of all conic sections?

Solution by T. P. Bhaskara Shastri and N. Durairajan.

The differential equation should be
$$\left(\frac{d^2y}{dx^2}\right)^2\cdot\frac{d^5y}{dx^5}-5\frac{d^2y}{dx^2}\cdot\frac{d^3y}{dx^3}\cdot\frac{d^4y}{dx^4}+\frac{40}{9}\left(\frac{d^3y}{dx^3}\right)^3=0.$$

The following method is indicated in Examples 49 and 50, Forsyth, p. 497.

Put $\frac{d^2y}{dx^2}=z$ and multiply the equation by z^n as an integrating actor : we have

$$z^{n+2}\frac{d^3z}{dx^3}-5z^{n+1}\frac{dz}{dx}\cdot\frac{d^2z}{dx^2}+\frac{40}{9}\cdot z^n\left(\frac{dz}{dx}\right)^3=0.$$

i.e.
$$\frac{d}{dx}\left[z^{n+2}\frac{d^2z}{dx^2}\right]-\frac{d}{dx}\left[\frac{n+7}{2}\cdot z^{n+1}\left(\frac{dz}{dx}\right)^2\right]$$
$$+\frac{(n+1)(n+7)}{2}z^n\left(\frac{dz}{dx}\right)^3+\frac{40}{9}\cdot z^n\left(\frac{dz}{dx}\right)^3=0.$$

The equation is integrable if we choose n so as to satisfy the equation

$$\frac{(n+1)(n+7)}{2}+\frac{40}{9}=0,$$

which gives $n=-\frac{11}{3}$ or $-\frac{13}{3}$.

Taking $n=-\frac{11}{3}$, a first integral of the equation is

$$z^{-\frac{5}{3}}\frac{d^2z}{dx^2}-\frac{5}{3}\cdot z^{-\frac{8}{3}}\left(\frac{dz}{dx}\right)^2=\text{a constant.}$$

Integrating,
$$z^{-\frac{5}{3}}\frac{dz}{dx}=\mathrm{A}x+\mathrm{B}.$$

Integrating again

$$-3\,z^{-\frac{2}{3}}=\mathrm{A}\,x^2+2\mathrm{B}x+2\mathrm{C}.$$

Thus we get an equation of the form

$$\frac{d^2y}{dx^2}=(a+2bx+cx^2)^{-\frac{3}{2}}$$

where a, b, c, are arbitrary constants.

Put $\mathrm{G}=y(a+2bx+cx^2)^{-\frac{1}{2}}$ or $y\mathrm{X}^{-\frac{1}{2}}$

The equation reduces to the form

$$\mathrm{X}^2\frac{d^2\mathrm{G}}{dx^2}+2\mathrm{X}(b+cx)\frac{d\mathrm{G}}{dx}=1+\mathrm{G}(b^2-ac).$$

$$\therefore\quad \frac{d}{dx}\left[\mathrm{X}^2\left(\frac{d\mathrm{G}}{dx}\right)^2\right]=[2+2\mathrm{G}\,(b^2-ac)]\frac{d\mathrm{G}}{dx}.$$

$$\therefore\quad \mathrm{X}^2\left(\frac{d\mathrm{G}}{dx}\right)^2=d+2\mathrm{G}+\mathrm{G}^2(b^2-ac).$$

$$\therefore\quad \int\frac{d\mathrm{G}}{\{d+2\mathrm{G}+\mathrm{G}^2(b^2-ac)\}^{\frac{1}{2}}}=e'+\int\frac{dx}{(a+2bx+cx^2)}.$$

Taking the case when b^2-ac is positive, the primitive is

$$\frac{1}{\sqrt{b^2-ac}}\log\left\{(b^2-ac)G+2+\sqrt{(b^2-ac)(d+2G+G^2\,\overline{b^2-ac})}\right\}$$

$$=2\frac{1}{\sqrt{b^2-ac}}\log.\ e.\frac{cx+b-\sqrt{b^2-ac}}{cx+b+\sqrt{b^2-ac}};$$

i.e. $$\left[(b^2-ac)G+2+\sqrt{(b^2-ac)(d+2G+G^2.\overline{b^2-ac})}\right]^2$$

$$=e.\frac{cx+b-\sqrt{b^2-ac}}{cx+b+\sqrt{b^2-ac}};$$

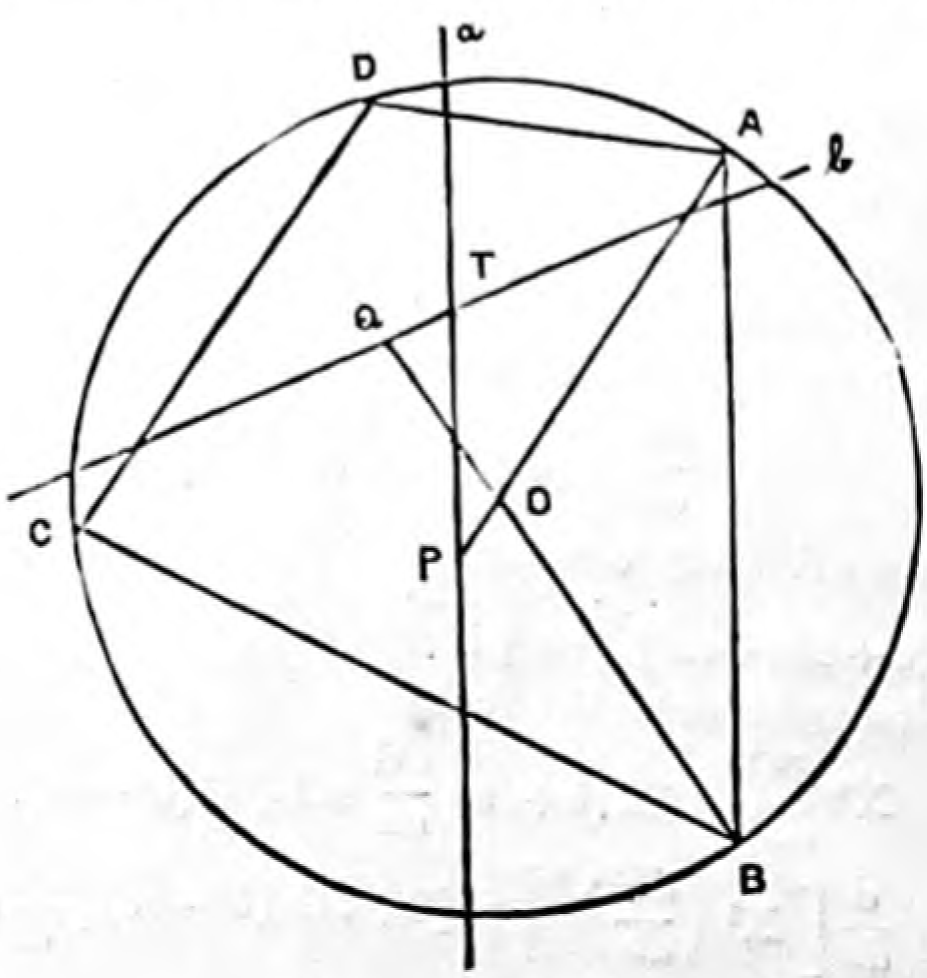
Rationalising and substituting for G, we get the primitive which contains the five arbitrary constants a, b, c, d and e.

Question 520.

(R. VYTHYANATHASWAMY) :—ABCD is a quadrilateral inscribed in a circle, centre O. Denoting the simson line of A *w.r.t* BCD by a and so on, shew that if, a, b, c, d intersect OA, OB, OC, OD in P, Q, R, S, and themselves cointersect at T, then P, Q, R, S, T, O are concyclic.

Remarks by N. Sankara Aiyar, and V. V. S. Narayan.

If, as the question states, P, Q, T, O are cyclic, then



$$P\hat{T}Q=P\hat{O}Q=A\hat{O}B.$$

But $P\hat{T}Q$ is easily to be the supplement of $A\hat{O}B$. Hence the question is incorrect,

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Question 542.

(K. V. Anantanarayana Sastri, B. A.) :—Expand $\theta \cot \frac{1}{4}\theta$ in powers of $\cos\theta$.

Solution by K. J. Sanjana and T. P. Trivedi.

In the expansion

$$\frac{\text{vers}^{-1}x}{\sqrt{(2x)}}=1+\frac{1}{3}\,\frac{x}{4}+\frac{1\cdot 3}{5}\,\frac{x^2}{4^2\lfloor 2}+\frac{1\cdot 3\cdot 5}{7}\,\frac{x^3}{4^3\lfloor 3}+\dots,$$

put $\text{vers}^{-1}x=\theta$; then $x=1-\cos\theta$, $\sqrt{(2x)}=2\sin\frac{\theta}{2}$, and we get

$$\frac{\theta}{2\sin\frac{\theta}{2}}=1+\frac{1}{3}\,\frac{(1-\cos\theta)}{4\lfloor 1}+\frac{1\cdot 3}{5}\,\frac{(1-\cos\theta)^2}{4^2\lfloor 2}+\dots\dots$$

Again, in the identity,

$$\theta\cot\theta=1-\frac{\sin^2\theta}{3}-\frac{2}{3}\,\frac{\sin^4\theta}{5}-\frac{2\cdot 4}{3\cdot 5}\,\frac{\sin^6\theta}{7}-\dots,$$

change θ to $\frac{1}{2}\theta$; as $\sin^2\frac{1}{2}\theta=\frac{1}{2}(1-\cos\theta)$, we get

$$\frac{\theta}{2}\cot\frac{\theta}{2}=1-\frac{1}{3}\,\frac{(1-\cos\theta)}{2}-\frac{2}{3}\,\frac{(1-\cos\theta)^2}{2^2\cdot 5}-\dots\dots$$

Adding up the two results, the left side gives $\frac{\theta}{2}\left(\text{cosec}\,\frac{\theta}{2}+\cot\frac{\theta}{2}\right)$

i. e., $\frac{\theta}{2}\cot\frac{\theta}{4}$; the right side gives

$$2-\frac{(1-\cos\theta)}{3\cdot 2}\left\{\frac{1}{1}-\frac{1}{2\lfloor 1}\right\}-\frac{(1-\cos\theta)^2}{5\cdot 2^2}\left\{\frac{2}{3}-\frac{1\cdot 3}{2^2\cdot 2}\right\}$$
$$-\frac{(1-\cos\theta)^3}{7\cdot 2^3}\left\{\frac{2\cdot 4}{3\cdot 5}-\frac{1\cdot 3\cdot 5}{2^3\lfloor 3}\right\}\dots\dots$$

Twice this series is therefore the required expansion.

Question 543.

(R. Vythynathaswamy) :—Shew that

$$1^r+2^r+3^r+\dots n^r=(c_2\Delta+c_3\Delta^2+\dots c_{r+1}\Delta^r)0^r$$

where $c_p={}^{n+1}C_p$.

Solution by R. J. Pocock, B.A., B.Sc., F.R.A.S., and A. Narasinga Rao.

We have the general theorem

$$U+U_1+U_2+\dots U_n=(1+E+E^2+\dots E^n)U_0.$$
$$=\frac{E^{n+1}-1}{E-1}U_0=\frac{(1+\Delta)^{n+1}-1}{\Delta}U_0.$$
$$=(n+1)U_0+\frac{n(n+1)}{2\,!}\Delta U_0+\dots$$
$$=(c_1+c_2\Delta+c_3\Delta^2+\dots)U_0.$$

Putting $U_0=0^r$, we have the required result remembering that if $n>r$ all terms after $\Delta^r.0^r$ vanish.

Question 544.

(SELECTED) :—Evaluate the definite integrals

$$\int_0^\infty e^{-x^2} \sin ax^2 \, dx ; \int_0^\infty e^{-x^2} \cos ax^2 \, dx.$$

Solution (1) *by R. J. Pocock B. A., B. Sc., F.R.A.S., T. P. Trivedi, M. A., LL. B., and K. J. Sanjana, M. A.* (2) *by J. M. Bose, M.A., B.Sc.*

(1) Let $P=\int_0^\infty e^{-x^2} \cos ax^2 dx$, $Q=\int_0^\infty e^{-x^2} \sin ax^2 dx.$

Then

$$P-Q\,i=\int_0^\infty e^{-x^2}(\cos ax^2-i \sin ax^2)dx$$

$$=\int_0^\infty e^{-x^2(1+ai)} dx$$

$$=\frac{1}{2}\sqrt{\frac{\pi}{1+ai}}=\frac{\sqrt{\pi}}{2(1+a^2)^{\frac{1}{2}}}(1-ai)^{\frac{1}{2}}$$

$$=\frac{\sqrt{\pi}}{2(1+a^2)^{\frac{1}{2}}}\left(\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}\right), \text{ where } \tan\theta=a.$$

Hence $P=\frac{1}{2}\left(\frac{\pi}{1+a^2}\right)^{\frac{1}{2}}\cos\frac{\theta}{2}=\frac{1}{2\sqrt{2}}\left\{\frac{\pi}{1+a^2}\left(1+\sqrt{1+a^2}\right)\right\}^{\frac{1}{2}},$

and $Q=\frac{1}{2}\left(\frac{\pi}{1+a^2}\right)^{\frac{1}{2}}\sin\frac{\theta}{2}=\frac{1}{2\sqrt{2}}\left\{\frac{\pi}{1+a^2}\left(\sqrt{1+a^2}-1\right)\right\}^{\frac{1}{2}}.$

(2) We have $\int_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$

Consider the integral $\int e^{-z^2} dz$, taken round a contour consisting of the real axis OA, an arc of a circle AB and a line OB inclined to OA at an angle α.

Since the function has no singularities within the contour

$$\int_C=\int_{OA}+\int_{AB}+\int_{BC}=0.$$

If OA=R, then at all points of OA z is real; hence

$$\int_{OA} e^{-z^2}dz=\int_0^R e^{-x^2}dx=\frac{\sqrt{\pi}}{2} \text{ when R is infinite.}$$

At all points on AB, we may put $z=R(\cos\theta+i\sin\theta)$ where R is constant and θ varies from θ to α.

$$\therefore \quad \int_{AB} = iR\int_{o}^{\alpha} e^{-R^2(\cos 2\theta + i \sin 2\theta)} \; (i \sin\theta + \cos\theta) d\theta$$

$$= iR\int_{o}^{\alpha} e^{-R^2 \cos 2\theta} . e^{i(\theta - R^2 \sin 2\theta)} d\theta.$$

At all points within the range of integration the real and imaginary parts of

$$\left| R \int e^{-R^2 \cos 2\theta} \; e^{i(\theta - R^2 \sin 2\theta)} \, d\theta \right| < \left| R \int e^{-R^2 \cos 2\theta} d\theta \right|.$$

Let $\alpha < \frac{\pi}{4}$ and consider

$$R \int_{o}^{\frac{\pi}{4}} e^{-R^2 \cos 2\theta} d\theta = R \left[\int_{o}^{\frac{1}{8}\pi} + \int_{\frac{1}{8}\pi}^{\frac{1}{4}\pi} \right]$$

Within the range $0 \leq \theta \leq \frac{\pi}{8}$, we have $\cos 2\theta \geq \frac{1}{\sqrt{2}}$

$$R \left| e^{-R^2 \cos 2\theta} \right| \leq R \left| e^{-R^2 / \sqrt{2}} \right|$$

which shows that the first integral within the bracket tends to the limit zero when R is indefinitely increased.

Similarly with in the range $\frac{\pi}{8} \leq \theta \leq \frac{\pi}{4}$, $\sqrt{2} \sin 2\theta \geq 1$.

$$\therefore \; R\sqrt{2}R \left| e^{-R^2 \cos 2\theta} \sin 2\theta \right| \geq R \left| e^{-R^2 \cos 2\theta} \right|$$

$$\therefore \left| \frac{R\sqrt{2}}{2} \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} e^{-R^2 \cos 2\theta} \, d\,(\cos 2\theta \right| > \left| R \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} e^{-R^2 \cos 2\theta} d\theta \right|$$

which shows that the second integral is also zero when $R = \infty$.

Hence
$$\int_{AB} = 0.$$

and
$$\frac{\sqrt{\pi}}{2} \int_{BO} e^{-z^2} dz = 0;$$

at all points on BO put $z = R(\cos \alpha + i \sin \alpha)$ where R varies from ∞ to zero.

$$\therefore \; \frac{\sqrt{\pi}}{2} - \int_{o}^{\infty} e^{-R^2(\cos 2\alpha + \sin 2\alpha)} dR (\cos \alpha + i \sin \alpha) = 0.$$

$$\therefore \; \int_{o}^{\infty} e^{-R^2 (\cos 2\alpha + i \sin 2\alpha)} dR = \frac{\sqrt{\pi}}{2} (\cos \alpha - i \sin \alpha).$$

Equating real and imaginary parts

$$\int e^{-R^2 \cos 2\alpha} \cos (R^2 \sin 2\alpha)\, dR = \frac{\sqrt{\pi} \cos \alpha}{2},$$

$$\int e^{-R \cos 2\alpha} \sin (R^2 \sin 2\alpha)\, dR = \frac{\sqrt{\pi} \sin \alpha}{2}.$$

Since R is real, put

$$r^2 \sin 2\alpha = ax^2$$
$$r^2 \cos 2\alpha = x^2.$$

Hence $\tan 2\alpha = a$, so that $\alpha < \frac{\pi}{4}$ so long as a is finite.

Thus
$$\int_0^{\infty} e^{-x^2} \cos ax^2 dx = \frac{\sqrt{\pi}}{2} \frac{\cos \alpha}{(1+a^2)^{\frac{1}{4}}}$$

$$\int_0 e^{-x^2} \sin ax^2 dx = \frac{\sqrt{\pi}}{2} \frac{\sin \alpha}{(1+a^2)^{\frac{1}{4}}}.$$

Question 547.

(V. K. Aravamudan)) :—Prove that the oblique trajectories of the system of curves represented by the elimination of t between

$$x = f(t, a)\ ;\ y = \phi(t, a)$$

a being the parameter of the family, are given by replacing a by the complete integral of

$$n\left\{\left(\frac{\partial f}{dt}\right)^2 + \left(\frac{\partial \phi}{dt}\right)^2\right\} + \frac{\partial a}{\partial t}\left\{ n\left[\frac{\partial \phi}{dt}\cdot\frac{\partial \phi}{\partial a} + \frac{\partial f}{\partial t}\cdot\frac{\partial f}{\partial a}\right] - \frac{\partial \phi}{\partial t}\cdot\frac{\partial f}{\partial a} + \frac{\partial f}{\partial t}\cdot\frac{\partial \phi}{\partial a}\right\} = 0,$$

and hence find the trajectories of a system of confocal ellipses, n being equal to the tangent of the angle of intersection.

Solution by T. P. Bhaskara Shastri and N. Durairajan.

Let the equation of the trajectory be

$$\xi = f(t, a).\ \eta = \phi(t, a) \quad \ldots \quad \ldots \quad \ldots (1)$$

where a is to be considered an unknown function of ξ and η to be determined so that the curve may be the trajectory. We now have

$$\frac{dy}{dx} = \frac{\frac{\partial \phi}{\partial t}}{\frac{\partial f}{\partial t}};\ \frac{d\eta}{d\xi} = \frac{\frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial a}\cdot\frac{\partial a}{\partial t}}{\frac{\partial f}{\partial t} + \frac{\partial f}{\partial a}\cdot\frac{\partial a}{\partial t}} \quad \ldots \quad \ldots (2)$$

and n
$$=\frac{\frac{dy}{dn}-\frac{d\eta}{d\xi}}{1+\frac{dy}{dx}\cdot\frac{dy}{d\xi}}=\frac{\frac{\partial\phi}{\partial t}\left(\frac{\partial\phi}{\partial t}+\frac{\partial f}{\partial a}\cdot\frac{\partial a}{\partial t}\right)-\frac{\partial f}{\partial t}\left(\frac{\partial\phi}{\partial t}+\frac{\partial\phi}{\partial a}\cdot\frac{\partial a}{\partial t}\right)}{\frac{\partial j}{\partial}\left(\frac{\partial f}{\partial t}+\frac{\partial f}{\partial a}\cdot\frac{\partial a}{\partial t}\right)+\frac{\partial\phi}{\partial t}\left(\frac{\partial\phi}{\partial t}+\frac{\partial\phi}{\partial a}\cdot\frac{\partial a}{\partial t}\right)}.$$

$$\therefore\quad n\left\{\left(\frac{\partial f}{\partial t}\right)^2+\left(\frac{\partial\phi}{\partial t}\right)^2\right\}+n\left\{\frac{\partial f}{\partial t}\cdot\frac{\partial f}{\partial a}\cdot\frac{\partial a}{\partial t}+\frac{\partial\phi}{\partial t}\cdot\frac{\partial\phi}{\partial a}\cdot\frac{\partial a}{\partial t}\right\}$$
$$-\frac{\partial a}{\partial t}\left\{\frac{\partial\phi}{\partial t}\cdot\frac{\partial f}{\partial a}-\frac{\partial f}{\partial t}\cdot\frac{\partial\phi}{\partial a}\right\}=0.$$

$$\therefore\quad n\left\{\left(\frac{\partial f}{\partial t}\right)^2+\left(\frac{\partial\phi}{\partial t}\right)^2\right\}+\frac{\partial a}{\partial t}\left\{n\left[\frac{\partial\phi}{\partial t}\cdot\frac{\partial\phi}{\partial a}+\frac{\partial f}{\partial t}\cdot\frac{\partial f}{\partial a}\right]\right.$$
$$\left.-\frac{\partial\phi}{\partial t}\cdot\frac{\partial f}{\partial a}+\frac{\partial f}{\partial t}\cdot\frac{\partial\phi}{\partial a}\right\}=0.$$

Replacing a by the complete integral of this in equations(**1**), we get the equations of the oblique trajectories.

Let the equation of the system of confocal ellipses be

$$x=\sqrt{a^2+\lambda}\cos t\qquad y=\sqrt{b^2+\lambda}\sin t.$$

Then, by the preceding part of the problem, the oblique trajectories are given by replacing λ by the integral of

$$n\left\{(a^2+\lambda)\sin^2 t+(b^2+\lambda)\cos^2 t\right\}+\frac{d\lambda}{dt}\left\{\frac{1}{2}\left[-\frac{\sqrt{b^2+\lambda}}{\sqrt{a^2+\lambda}}\cos^2 t\right.\right.$$
$$\left.\left.-\frac{\sqrt{a^2+\lambda}}{\sqrt{b^2+\lambda}}\sin^2 t\right]\right\}=0,$$

i.e. $$n\left\{a^2\sin^2 t+b^2\cos^2 t+\lambda\right\}=\frac{1}{2}\frac{d\lambda}{dt}\sqrt{\frac{a^2\sin^2 t+b^2\cos^2 t+\lambda}{(a^2+\lambda)(b^2+\lambda)}}\Bigg],$$

$$\therefore\quad 2ndt=\frac{d\lambda}{\sqrt{(a^2+\lambda)(b^2+\lambda)}}\quad\therefore\quad 2n(t+A)=\cosh^{-1}\frac{\lambda+\frac{a^2+b^2}{2}}{\frac{a^2-b^2}{2}}.$$

$$\therefore\qquad \lambda=\frac{a^2-b^2}{2}.\cosh_2 n(t+A)-\frac{a^2+b^2}{2}.$$

Replacing this value of λ, we get for the trajectories

$$\xi=\sqrt{\frac{a^2-b^2}{2}}\cdot\sqrt{\cosh 2n(t+A)+1}\cos t.$$

$$\eta=\sqrt{\frac{a^2-b^2}{2}}\sqrt{\cosh 2n(t+A)-1}\sin t.$$

which are easily reduced to the form

$$\xi = c \cos t \cosh n(t+A)$$
$$\eta = c \sin t \sinh n(t+A),$$

where $2c$ is the distance between the foci. (Forsyth : p. 143.)

Question 551.

(T. P. Trivedi, M.A., LL.B.) :—Prove that

$$\iiint (abcdfghlmn)(xyz)^2\, dx\, dy\, dz$$

taken throughout a tetrahedron bounded by the co-ordinate planes and the plane $x/p+y/q+z/r=1$, is equal to

$$\frac{pqr}{60}\{(ap^2+bq^2+cr^2+fqr+grp+hpq)+5(lp+mq+nr)+10d\}.$$

Solution by R. J. Pocock, B.A., B.Sc., F.R.A.S., and R. Vythynathaswamy.

Each term of the Integral is a Dirichlet Integral.

Thus $\iiint ax^2\, dx\, dy\, dz = \frac{ap^3qr\Gamma(3)}{\Gamma(5)}\int_0^1 Z^4 dZ = \frac{ap^3qr}{60}.$

$$\iiint 2fyz\, dx\, dy\, dz = \frac{2f\cdot p\cdot q^2\cdot r^2\cdot\Gamma(2)\Gamma(2)}{\Gamma(5)}\int_0^1 Z^4 dZ = \frac{f\cdot p\cdot q^2\cdot r^2}{60}.$$

$$\iiint 2lx\, dx\, dy\, dz = \frac{2\cdot l\cdot p^2\cdot q\cdot r\cdot\Gamma(2)}{\Gamma(4)}\int_0^1 Z^3 dZ = \frac{5lp^2qr}{60}.$$

$$\iiint d\cdot dx\, dy\, dz = d\cdot p\cdot q\cdot r\,\frac{1}{\Gamma(3)}\cdot\int_0^1 Z^2 dZ = \frac{d\cdot p\cdot q\cdot r}{6}.$$

The given Integral is therefore equal to

$$\frac{pqr}{60}\{(ap^2+bq^2+cr^2+fqr+grp+hpq)+5(lp+mq+nr)+10d\}.$$

QUESTIONS FOR SOLUTION.

592. (S. Narayana Aiyar, M.A.) :—Show that a^n is equal to

$$\begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & (1+a) \\
1 & 2 & 1 & 0 & 0 & 0 & 0 & (1+a)^2 \\
1 & 3 & 3 & 1 & 0 & 0 & 0 & (1+a)^3 \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & & \cdots \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & & \cdots \\
1 & {}_{n-3}C_1 & {}_{n-3}C_2 & {}_{n-3}C_3 & 1 & 0 & 0 & (1+a)^{n-3} \\
1 & {}_{n-2}C_1 & {}_{n-2}C_2 & {}_{n-2}C_3 & n-2 & 1 & 0 & (1+a)^{n-2} \\
1 & {}_{n-1}C_1 & {}_{n-1}C_2 & {}_{n-1}C_3 & {}_{n-1}C_{n-3} & n-1 & 1 & (1+a)^{n-1} \\
1 & {}_{n}C_1 & {}_{n}C_2 & {}_{n}C_3 & {}_{n}C_3 & {}_{n}C_2 & n & (1+a)^n
\end{vmatrix}$$

593. (S. Narayana Aiyar, M.A.) :—Shew that

$$\frac{\dfrac{1}{a}+\dfrac{x}{a+1}+\dfrac{x^2}{1\cdot 2}\cdot\dfrac{1}{a+2}+\dfrac{x^3}{1\cdot 2\cdot 3}\cdot\dfrac{1}{a+3}+\ldots \text{ to } \infty}{\dfrac{1}{a+1}+\dfrac{x}{a+2}+\dfrac{x^2}{1\cdot 2}\dfrac{1}{a+3}+\dfrac{x^3}{1\cdot 2\cdot 3}\cdot\dfrac{1}{a+4}+\ldots \text{to } \infty} \text{ is equal to}$$

$$\frac{\dfrac{1}{a}-\dfrac{x}{a(a+1)}+\dfrac{x^2}{a(a+1)(a+2)}-\dfrac{x^3}{a(a+1)(a+2)(a+3)}+\ldots\text{to } \infty}{\dfrac{1}{a+1}-\dfrac{x}{(a+1)(a+2)}+\dfrac{x^2}{(a+1)(a+2)(a+3)}-\ldots\text{to } \infty}.$$

594. (A. A. Krishnaswami Aiyangar, B.A.) :—Establish the following approximate formula for the length of a circular arc :

$$\frac{32768R-5376Q+168H-C}{2735}$$

where C is the chord of the whole arc

H ... half the arc

Q ... quarter the arc

R ... eighth of the arc.

595. (A. A. Krishnaswami Aiyangar, B. A.) :—ABC is a triangle. D, E, F are the points of contact of the in-circle with the sides BC, CA, AB. D′, E′, F′ are the middle points of AI, BI, CI where I is the incentre. Show that the circles on DD′, EE′, FF′ as diameters are concurrent at the in-Feurbach point. Also prove that a similar property exists for the other Feurbach points.

596. (N. Sankara Aiyar, M.A.) :—Show that

$$\frac{1}{2}\cdot\frac{1}{2}-\frac{1\cdot3}{2\cdot4^2}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6^2}-\ldots\ldots=\log\frac{1+\sqrt{2}}{2},$$

and

$$\frac{1}{2}\cdot\frac{1}{2}+\frac{1\cdot3}{2\cdot4^2}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6^2}+\ldots\ldots=\log 2.$$

597. (A. Narasinga Rao) :—Shew that the parabola is the only curve such that the triangle formed by any three points on the curve is proportional to the triangle formed by the tangents at these points.

598. (A, Narasinga Rao) :—Arrange 13 points so as to form 18 rows of 3 each, no row containing more than 3.

599. (V. V. S. Narayan) :—If two circles cut orthagonally find the locus of a point such that the tangents from it to the two circles form a harmonic pencil.

600. (V. V. S. Narayan) :—If PQ is a diameter of the circumcircle of a triangle ABC, shew that the axes of the inscribed parabolas whose foci are P and Q intersect at a point R on the circle. Shew further that the axis of the inscribed parabola whose focus is R is perpendicular to PQ.

601. (S. Narayanan, B.A., L.T.) :—If X is the Steiner's envelope of a given triangle and Y, the inconic having its centre at the circumcentre, shew that (1) X and Y touch the sides of the triangle in the same three points ; (2) the other six points of intersection of X with the sides of the triangle lie on a conic Z ; (3) Z is concentric with and homethetic to the polar reciprocal of Y with respect to the polar circle of the triangle.

602. (S. Krishnaswami Iyengar) :—A variable straight line meets two fixed straight lines intersecting at O in the points P and Q. If OP, OQ subtend equal angles at a fixed point A, show that PQ passes through a fixed point B such that BAO is a right angle.

603. (S. KRISHNASWAMI IYENGAR) :—Shew that

$$1-(x)^2+\left(\frac{x^2}{2!}\right)^2-\left(\frac{x^3}{3!}\right)^2+\ldots\ldots=\left(\frac{2}{\pi}\right)\int_0^\pi \cos^2(x\sin\theta)\,d\theta-1.$$

and find the value of

$$1+x^2+\left(\frac{x^2}{2!}\right)^2+\left(\frac{x^3}{3!}\right)^2+\ldots\ldots$$

604. (R. VYTHYNATHSWAMY) :—A quadrilateral PQRS has its vertices on a conicoid. Shew that the plane through PQ which is the harmonic conjugate of the central plane through PQ with respect to the planes PQR, PQS is parallel to the corresponding plane in the case of RS.

605. (S. RAMANUJAN) :—Shew that, when $x=\infty$,

$$\frac{(x+a-b)!\;(8x+2b)!\;(9x+a+b)!}{(3x+a-c)!\;(3x+a-b+c)!\;(12x+3b)!}=\sqrt{\frac{2}{3}}.$$

606. S. RAMANUJAN) :—Shew that

$$\sum_0^\infty\left\{\frac{(\sqrt{5}-2)^{2n+1}}{(2n+1)^2}\right\}=\frac{\pi^2}{24}-\frac{1}{12}\left\{\log\,(2+\sqrt{5})\right\}^2.$$

607. (K. J. SANJANA, M.A.) :—Shew how to solve the differential equations

$$\frac{\partial^2 u}{\partial y\,\partial x}-\frac{\partial^2 u}{\partial y\,\partial z}=u(x^2-z^2)\phi(x,y,z),$$
$$\frac{\partial^2 u}{\partial z\,\partial y}-\frac{\partial^2 u}{\partial x\,\partial z}=u(y^2-z^2)\psi(x,y,z),$$
$$\frac{\partial^2 u}{\partial x\,\partial y}+\frac{\partial^2 u}{\partial y\,\partial z}+\frac{\partial^2 u}{\partial x\,\partial z}=u\theta(x,y,z),$$

where ϕ, ψ, θ denote given functions of x, y and z.

Solve in particular the case when

$$\phi(x,y,z)=\psi(x,y,z)=3/(x^3+y^3+z^3),$$

and $$\theta(x,y,z)=3(x^3+y^3+z^3+2x^2+2y^2+2z^2)(x^3+y^3+z^3).$$

608. (M. BHIMASENA RAO) :—Shew that

(i) $\left(1+e^{-\pi}\right)\left(1+e^{-3\pi}\right)\left(1+e^{-5\pi}\right)\ldots\ldots=2^{\frac{1}{4}}e^{-\frac{\pi}{24}}$

(ii) $\left(1+e^{-\pi\sqrt{3}}\right)\left(1+e^{-3\pi\sqrt{3}}\right)\left(1+e^{-5\pi\sqrt{3}}\right)\ldots=2^{\frac{1}{3}}e^{-\frac{\pi\sqrt{3}}{24}}$.

(iii) $\left(1+e^{-\pi\sqrt{7}}\right)\left(1+e^{-3\pi\sqrt{7}}\right)\ldots\ldots\ldots=2^{\frac{1}{2}}e^{-\frac{\pi\sqrt{7}}{24}}$.

609. (M. Bhimasena Rao) :—Shew that

$$\text{(i)}\quad \frac{1}{e^{2\pi}-1}+\frac{2^{4n+1}}{e^{4\pi}-1}+\frac{3^{4n+1}}{e^{6\pi}-1}+\frac{4^{4n+1}}{e^{8\pi}-1}+\dots\dots\dots=\frac{B_{2n+1}}{4(2n+1)}$$

$$\text{(ii)}\quad \frac{1}{e^{\pi}+1}+\frac{3^{4n+1}}{e^{3\pi}+1}+\frac{5^{4n+1}}{e^{5\pi}+1}+\dots=\frac{B_{2n+1}}{4(2n+1)}(2^{4n+1}-1),$$

where B_1, B_2, B_3,...are the numbers of Bernoulli and n is any positive integer in (i), and zero or any positive integer in (ii).

610. (K. V. Ananthanarayanasastri, B.A.) :—If T is a point on the directrix of a parabola and t_1, t_2 the lengths of the tangents TP, TQ from it to the parabola, shew that the length of the arc PQ of the parabola is

$$\frac{t_1^3+t_2^3}{t_1^2+t_2^2}+\frac{t_1^2t_2^2}{(t_1^2+t_2^2)^{\frac{3}{2}}}\left\{\sinh^{-1}\frac{t_1}{t_2}+\sinh^{-1}\frac{t_2}{t_1}\right\}.$$

611. (T. P. Trivedi, M.A., LL.B.) :—Prove that the quadrature of the ellipsoid may be expressed in the form

$$4\pi ab\left\{1-\frac{e^2+e'^2}{6}-S\right\}$$

where e and e' are the eccentricities of the sections by the planes $x=0$ and $y=0$ and

$$S=\sum\frac{e^{2m}\,e'^{2m'}}{(2m+2m'+1)(2m+2m'-1)}\,\frac{(2m)!\,(2m')!}{(m!\,m'!)^2\,4^{m+m'}}$$

612. (N. Ganapathi Subba Aiyar) :—A series of ellipses have a common focus, pass through a given point and have their major axes of the same length. Find their envelope.